Nonlinear massive spin–two field generated by higher derivative gravity

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Abstract

We present a systematic exposition of the Lagrangian field theory for the massive spin–two field generated in higher–derivative gravity upon reduction to a second–order theory by means of the appropriate Legendre transformation. It has been noticed by various authors that this nonlinear field overcomes the well known inconsistency of the theory for a linear massive spin–two field interacting with Einstein's gravity. Starting from a Lagrangian quadratically depending on the Ricci tensor of the metric, we explore the two possible second–order pictures usually called "(Helmholtz–)Jordan frame" and "Einstein frame". In spite of their mathematical equivalence, the two frames have different structural properties: in Einstein frame, the spin–two field is minimally coupled to gravity, while in the other frame it is necessarily coupled to the curvature, without a separate kinetic term. We prove that the theory admits a unique and linearly stable ground state solution, and that the equations of motion are consistent, showing that these results can be obtained independently in either frame (each frame therefore provides a self–contained theory). The full equations of motion and the (variational) energy–momentum tensor for the spin–two field in Einstein frame are given, and a simple but nontrivial exact solution to these equations is found. The comparison of the energy–momentum tensors for the spin–two field in the two frames suggests that the Einstein frame is physically more acceptable. We point out that the energy–momentum tensor generated by the Lagrangian of the linearized theory is unrelated to the corresponding tensor of the full theory. It is then argued that the ghost–like nature of the nonlinear spin–two field, found long ago in the linear approximation, may not be so harmful to classical stability issues, as has been expected.

1 Introduction

A consistent theory of a gravitationally interacting spin–two field could not be developed until a significant progress was made in an apparently unrelated subject, i.e. higher–derivative metric theories of gravity. It is well known that a single linear spin–two field cannot be consistently coupled to gravity. It is therefore a common belief that Nature avoids the consistency problem by simply not creating fundamental spin–two (nor higher spin) fields except gravity itself. Nevertheless the subject has remained fascinating over decades and some authors have studied various aspects of linear spin–two fields [1, 2, 3], in particular their dynamics in Einstein spaces [4, 5].

On the other hand, higher–derivative metric theories of gravity, where the Lagrangian is a scalar nonlinear function of the curvature tensor (hence in this paper they are named nonlinear gravity theories, NLG) have attracted much more attention. Most work was centered on quadratic theories, i.e. on Lagrangians being quadratic polynomials in the Ricci tensor and the curvature scalar [6, 7], but several authors studied more general Lagrangians [8]. These theories turned out to be inadequate as candidates for foundations of quantum gravity since they are non–unitary, but recently play a role as effective field theories. What is more relevant here, it was found that their particle spectrum contains a massive spin–two field. The dynamics of this field can be described and investigated by recasting the fourth-order NLG theory into a standard nonlinear second–order Lagrangian field theory. The procedure entails a decomposition of the dynamical data, consisting of the metric field $\tilde{g}_{\mu\nu}$ and its derivatives up to the third order, into a set of independent fields describing the physical state by their values and their first derivatives only. In this peculiar sense, one may say that the single "unifying" field $\tilde{g}_{\mu\nu}$ is replaced by (or decomposed into) a multiplet of gravitational fields.

An adequate mathematical tool for this purpose is provided by a specific Legendre transformation [9, 10, 11]. Although the transformation has been known for more than a decade, it is not currently used in a systematic way. Instead, most papers on the nonlinear spin–two field have employed various *ad hoc* tricks adjusted to quadratic Lagrangians [12, 13, 14] (actually equivalent to the Legendre transformation for this particular case), but such approach does not allow one to fully exhibit the structure of the theory.

Although the Legendre transformation is essentially unique, the various fields of the resulting multiplet can be given different physical interpretations; the different choices are traditionally called "frames"[1]. In general, the original "Jordan frame" (JF; the name is borrowed from scalar–tensor theories) consisting of only the unifying metric $\tilde{g}_{\mu\nu}$, can be transformed into frames including fields of definite spin in two ways. A first possibility

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[1]The use of the word "frame" in this sense should be deprecated, because it does not refer to the choice of a physical reference frame, but this abuse of terminology is now so universally adopted that we feel that trying to introduce here a more appropriate term, for instance "picture" as it is used in Quantum Mechanics, would only lead to confusion.

is that the field $\tilde{g}_{\mu\nu}$ remains the spacetime metric, now carrying only two d.o.f., while the other degrees of freedom (previously carried by its higher derivatives) are encoded into auxiliary (massive) fields of definite spin: this is the Helmholtz–Jordan frame (HJF). Alternatively, one introduces (via an appropriate redefinition of the Legendre transformation) a new spacetime metric $g_{\mu\nu}$, while the symmetric tensor $\tilde{g}_{\mu\nu}$ is decomposed into spin–2 and spin–0 fields, forming in this way the massive, non–geometric components of the gravitational multiplet; these variables form together the "Einstein frame" (EF). Both frames are dynamically equivalent and very similar on the level of the field equations: the equations of motion are second–order Lagrange field equations, and in each frame the corresponding spacetime metric satisfies Einstein's field equations[2], thus the theory looks like ordinary general relativity, with the non–geometric components of the multiplet acting as specific matter fields. The two frames differ however in the action integral. In HJF the spin–0 and spin–2 fields are nonminimally coupled to gravity (to curvature) while there are no kinetic terms for those fields in the Lagrangian; only the metric has the standard Einstein–Hilbert Lagrangian $\tilde{R}(\tilde{g})$. In consequence, propagation PDEs for the fields with spin zero and two arise from the action in a more involved way (through the variation of the metric tensor), so the theory in this frame cannot be obtained by minimal coupling to ordinary gravity of some additional fields already possessing a definite dynamics in a fixed background spacetime. Yet it is remarkable that in the EF variables the theory has fully standard form [9, 10]: one recovers the $R(g)\sqrt{-g}$ Lagrangian for the metric and universal kinetic terms for the spin–0 and spin–2 fields (independently of the form of the original Lagrangian $L(\tilde{g})$ in JF), only the potential part of the action being affected by the actual form of $L(\tilde{g})$. The EF variables are uniquely characterised by these features, while in HJF different *ad hoc* redefinitions of the variables can be intertwined with the Legendre transformation (the latter being itself sometimes disguised as a mere change of variables) [12, 13].

Though matematically equivalent, the two frames are physically inequivalent; the difference is most clearly visible while defining the energy since the latter is very sensitive to redefinitions of the spacetime metric [15]. Both mathematical similarity to ordinary general relativity and physical arguments indicate that the EF is physical [15]: in HJF the energy–momentum tensor is unphysical, being linear in both non–geometric fields, while in EF the stress–energy tensor for the scalar field has the standard form and that for the spin-two field seems also more acceptable than in HJF.

Anyhow, in both frames NLG theories provide a consistent description of a self–gravitating massive spin–two field. The field is necessarily nonlinear and in quantum theory it is ghostlike. The latter defect is inferred from the fact that in the linearized theory the Lagrangian of the field appears with the sign opposite to that for linearized Einstein gravity [16]. This fact is interpreted in classical theory as related to the occur-

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[2]Some authors seem instead to believe that Einstein equations can be obtained only after redefinition of the spacetime metric, i.e.in EF.

rence of excitations with negative energy for the field, and in consequence as a signal of instability of the theory. However, in this paper we show (an incomplete proof was given previously in [13]) that the ground state solution (vacuum) *is* classically stable at the linear level. This does not prove the stability of the vacuum state whenever nonlinear terms are taken into account: we stress, however, that all the above mentioned features usually advocated as signals of instability, are equally derived within the linear approximation. The problem of energy is more subtle: in HJF the variational energy–momentum tensor is evidently unphysical, while in EF the Lagrangian is highly nonlinear (not even polynomial), and we will show in sect.7 that the linear approximation tells rather little about the energy density of the exact theory. Hence, the massive spin–two field generated by NLG theories is still worth investigating in the framework of classical Lagrangian field theory.

Most previous works [12, 13, 14] were centered on the particle spectrum of the theory and dealt only with the action integrals, while less attention was paid to the field equations and the structure of the theory. The field equations in HJF were given in [13], but these authors regarded the field equations in EF as extremely involved and thus intractable, they only studied the case where the spin–two field is assumed to be proportional to the metric tensor and thus can be described by a scalar function (its trace). In consequence, the dynamical consistency of the theory has always been taken for granted since the fourth–order equations in JF are consistent.

In the present paper we systematically investigate the nonlinear spin–two field generated by a NLG theory with a quadratic Lagrangian (1) in the framework of classical field theory, employing a Legendre transformation. As is well known, a spin–two field may be mathematically represented by tensor variables of different rank and symmetry properties [17, 2]. Any NLG theory generates in a natural way a representation of the field in terms of a symmetric, second rank tensor $\psi_{\mu\nu} = \psi_{\nu\mu}$, and this representation will be employed in this work.

Although we study a concrete Lagrangian (this is motivated in sects. 2 and 3) we employ no tricks adjusted to it. The paper is self–contained and provides an (almost) full exposition of the subject. The first part describes the nonlinear spin–two and spin-zero fields in HJF. The particle content in this frame is well known. The main new results shown here are:

- the field equations for the metric and the other two fields for any spacetime dimension $d \geq 4$;

- the fact that dimension $d = 4$ is distinguished in that the scalar field is decoupled and can be easily removed from the theory (we do so in the rest of the paper since we are interested in the description of the spin–two field);

- the fact (sect. 3) that the resulting equations of motion for the spin–two field do not generate further constraints besides the five ensuring the purely spin–2

character of the field;

- the internal consistency of the theory in HJF, ensured by strong Noether conservation laws;

- the linear stability of the unique ground state solution representing flat spacetime and vanishing spin–two field, assessed by the fact that small perturbations form plane waves with constant amplitudes.

In sect. 4 we show that the two possible ways to obtain the massless limit of the spin–two field described in the previous section yield the same result, so the massless limit is well–defined and leads to the propagation equations for gravitational perturbations in a Ricci–flat spacetime.

The main thrust of the paper is its second part (sect.5 to 9), where we investigate the equations of motion in Einstein frame. Here most results are new.

- A generic presentation of the Lagrangian theory for the nonlinear spin–two field in Einstein frame is given in sect. 5: in order to better exhibit the structure of the theory in this frame, we consider a Lagrangian being an arbitrary function of the Ricci tensor of the original metric in JF. We explicitly give the equations of motion for the massive field in the generic case, an expression (highly nonlinear) for the full energy–momentum tensor $T_{\mu\nu}$ of the field and four differential constraints imposed on the field by its dynamics.

- This generic theory is then specialized in sect. 6 to the case of the particular Lagrangian of eq. (33), which in HJF ensured that the scalar field drops out. The previous, generic equations produce in this case a fifth, algebraic constraint, which together with those already found ensures that also in EF the massive field has five degrees of freedom and is purely spin–two.

- A unique, linearly stable ground state solution is then found (without any simplifying assumptions); clearly it corresponds via the Legendre transformation to the ground state in HJF. The spin–two field is then redefined to make it vanish in this state (and in vacuum in general). It turns out that the consistency and hyperbolicity problems in this frame, investigated in sect. 7, are harder that in HJF and should be studied perturbatively; we study them in the linearized theory. The Lagrangian is computed in the lowest order (quadratic) approximation around the ground state solution to show the ghostlike character of the spin–two field. A detailed comparison is made with the theory of the linear spin–two field, and the fact that the energy–momentum tensor of the nonlinear theory is not approximated by the energy–momentum tensor derived from the Wentzel Lagrangian is fully explained.

- Finally, contrary to the common belief that the full (nonlinear) system of equations of motion in EF are intractably involved, in sect. 8 we give a simple but nontrivial solution to them.

Conclusions are in sect. 9, and Lagrange field equations and the energy–momentum tensor in EF for the redefined spin–two field as well as some other useful formulae are contained in Appendix.

2 Equations of motion for the gravitational multiplet in Helmholtz–Jordan frame

We will investigate dynamical structure and particle content of a nonlinear gravity (NLG) theory using Legendre transformation method [9, 10, 11]. The starting point is a d–dimensional manifold M, $d \geq 4$ (later the dimensionality will be fixed to $d = 4$) endowed with a Lorentzian metric $\tilde{g}_{\mu\nu}$. The inverse contravariant metric tensor will be denoted by $\tilde{g}^{\mu\nu}$ and $\tilde{g}^{\mu\alpha}\tilde{g}_{\alpha\nu} = \delta^{\mu}_{\nu}$; we introduce this nonstandard notation for further purposes. One need not view $\tilde{g}_{\mu\nu}$ as a physical spacetime metric, actually whether $\tilde{g}_{\mu\nu}$ or its "canonically conjugate" momentum is the measurable quantity determining all spacetime distances in physical world should be determined only after a careful examination of the physical content of the theory, rather than prescribed *a priori*. Formally $\tilde{g}_{\mu\nu}$ plays both the role of a metric tensor on M and is a kind of unifying field which will be decomposed in a multiplet of fields with definite spins; pure gravity is described in terms of the fields with the metric being a component of the multiplet. In general dynamics for $\tilde{g}_{\mu\nu}$ is generated by a nonlinear Lagrangian density $L\sqrt{-\tilde{g}} = f(\tilde{g}_{\mu\nu}, \tilde{R}_{\alpha\beta\mu\nu})\sqrt{-\tilde{g}}$ where $\tilde{g} \equiv \det(\tilde{g}_{\mu\nu})$ and $\tilde{R}_{\alpha\beta\mu\nu}$ is the Riemann tensor for $\tilde{g}_{\mu\nu}$; f is any smooth (not necessarily analytic) scalar function. Except for Hilbert–Einstein and Euler–Poincaré topological invariant densities the resulting variational Lagrange equations are of fourth order. The Legendre transformation technique allows one to deal with fully generic Lagrangians; from the physical standpoint, however, there is no need to investigate complicate or generic Lagrangians. Firstly, in the bosonic sector of low energy field theory limit of string effective action one gets in the lowest approximation the Hilbert–Einstein Lagrangian plus terms quadratic in the curvature tensor. Secondly, to obtain an explicit form of field equations and to deal with them effectively one needs to invert the appropriate Legendre transformation and in a generic case this amounts to solving nonlinear matrix equations. Hindawi, Ovrut and Waldram [18] have given arguments that a generic NLG theory has eight degrees of freedom and the same particle spectrum as in the quadratic Lagrangian (1) below, the only known physical difference lies in the fact that in the generic case one expects multiple nontrival (i.e. different from flat spacetime) ground state solutions. This result can be also derived from the observation that after the Legendre transformation the kinetic terms in the resulting (Helmholtz)

Lagrangian are universal, and only the potential terms keep the trace of the original nonlinear Lagrangian. If the latter is a polynomial of order higher than two in the curvature tensor, the Legendre map is only locally invertible and this leads to multi-valued potentials, generating a ground state solution in each "branch"; yet the form of the potential could produce additional dynamical contraints, affecting the number of degrees of freedom, only in non–generic cases. The physically relevant Lagrangians in field theory depend quadratically on generalized velocities and then conjugate momenta are linear functions of the velocities. For both conceptual and practical purposes it is then sufficient to envisage a quadratic Lagrangian

$$L = \tilde{R} + a\tilde{R}^2 + b\tilde{R}_{\mu\nu}(\tilde{g})\tilde{R}^{\mu\nu}(\tilde{g}). \tag{1}$$

In principle one should also include the term $\tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\alpha\beta\mu\nu}$ (in four dimensions it can be eliminated via Gauss–Bonnet theorem), but the presence of Weyl tensor causes troubles: although formally the Legendre transformation formalism works well there are problems with providing appropriate propagation equations for the conjugate momentum and with physical interpretation (particle content) of the field. We therefore suppress Weyl tensor in the Lagrangian. The Lagrangian cannot be purely quadratic: it is known from the case of restricted NLG theories (Lagrangian depends solely on the curvature scalar, $L = f(\tilde{R})$) that the linear term $\tilde{R}$ is essential [15] and we will see that the same holds for Lagrangians explicitly depending on Ricci tensor $\tilde{R}_{\mu\nu}$. The coefficients a and b have dimension $[\text{length}]^2$; contrary to some claims in the literature there are no grounds to presume that they are of order $(\text{Planck length})^2$ unless the Lagrangian (1) arises from a more fundamental theory (e.g. string theory) where $\hbar$ is explicitly present. Otherwise in a pure gravity theory the only fundamental constants are c and G; then a and b need not be new fundamental constants, they are rather related to masses of the gravitational multiplet fields. Here we assume that the NLG theory with the Lagrangian (1) is an independent one, i.e. it inherits no features or relationships from a possible more fundamental theory.

As was mentioned in the Introduction, in the Legendre transformation one replaces the higher derivatives of the field $\tilde{g}_{\mu\nu}$ by additional fields. In this section we assume that the original field $\tilde{g}_{\mu\nu}$ keeps the role of the physical spacetime metric, and the self–gravitating spin–two field originates from the "conjugate momenta" to $\tilde{g}_{\mu\nu}$.

We recall that for a second–order Lagrangian such as (1) one should properly choose the quantities to be taken as generalized velocities to define, via a Legendre map, conjugate momenta [11]. One cannot, for instance, use the partial derivatives $\tilde{g}_{\mu\nu,\alpha\beta}$ as generalized velocities since for covariant Lagrangians, e.g. (1), the Legendre map cannot be inverted: the Hessian, being the determinant of a 100×100 matrix, vanishes,

$$\det\left(\frac{\partial^2 L}{\partial\tilde{g}_{\mu\nu,\alpha\beta}\partial\tilde{g}_{\lambda\sigma,\rho\tau}}\right) = 0. \tag{2}$$

General covariance indicates which linear combinations of $\tilde{g}_{\mu\nu,\alpha\beta}$ can be used as the velocities, i.e. with respect to which combinations the Lagrangian is regular (the Hessian does not vanish). Clearly this is Ricci tensor $\tilde{R}_{\mu\nu}$. The explicit use of generally covariant quantities in this approach is supported by the Wald's theorem [19, 20] that only a generally covariant theory may be a consistent theory of a spin–two field. Following [11], in order to decompose $\tilde{g}_{\mu\nu}$ into fields with definite spins, one makes Legendre transformations of the Lagrangian (1) with respect to the two irreducible components of $\tilde{R}_{\mu\nu}$: its trace $\tilde{R}$ and the traceless part $\tilde{S}_{\mu\nu}$. In terms of $\tilde{S}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{d}\tilde{R}\tilde{g}_{\mu\nu}$, $\tilde{S}_{\mu\nu}\tilde{g}^{\mu\nu} = 0$, the Lagrangian reads

$$L = \tilde{R} + (a + \frac{b}{d})\tilde{R}^2 + b\tilde{S}_{\mu\nu}\tilde{S}^{\mu\nu}, \tag{3}$$

one assumes $ad + b \neq 0$ and $b \neq 0$. One then defines a scalar and a tensor canonical momentum via corresponding Legendre transformations:

$$\chi + 1 \equiv \frac{\partial L}{\partial \tilde{R}}, \qquad \pi^{\mu\nu} \equiv \frac{\partial L}{\partial \tilde{S}_{\mu\nu}}; \tag{4}$$

it is convenient to identify $\partial L/\partial \tilde{R}$ with $\chi + 1$ rather than with χ alone. From (3):

$$\chi = 2(a + \frac{b}{d})\tilde{R} \qquad \text{and} \qquad \pi^{\mu\nu} = 2b\tilde{S}^{\mu\nu} \tag{5}$$

hence fields χ and $\pi^{\mu\nu}$ are dimensionless and $\pi^{\mu\nu}$ is traceless, $\pi^{\mu\nu}\tilde{g}_{\mu\nu} = 0$. The new triplet of field variables $\{\tilde{g}_{\mu\nu}, \chi, \pi^{\mu\nu}\}$ defines the Helmholtz–Jordan Frame (HJF).

Equations of motion for this frame arise as variational Lagrange equations from Helmholtz Lagrangian [21, 9, 10, 11]. First one constructs the Hamiltonian

$$H = \frac{\partial L}{\partial \tilde{R}}\tilde{R} + \frac{\partial L}{\partial \tilde{S}_{\mu\nu}}\tilde{S}_{\mu\nu} - L \tag{6}$$

expressed in terms of $\tilde{g}_{\mu\nu}$ and the canonical momenta, it reads

$$H = \frac{d}{4(ad + b)}\chi^2 + \frac{1}{4b}\pi^{\mu\nu}\pi_{\mu\nu}, \tag{7}$$

here $\pi_{\mu\nu} = \tilde{g}_{\mu\alpha}\tilde{g}_{\nu\beta}\pi^{\alpha\beta}$; all indices are raised and lowered with the aid of $\tilde{g}^{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. Next one evaluates Helmholtz Lagrangian defined as

$$L_H \equiv \frac{\partial L}{\partial \tilde{R}}\tilde{R}(\tilde{g}, \partial\tilde{g}, \partial^2\tilde{g}) + \frac{\partial L}{\partial \tilde{S}_{\mu\nu}}\tilde{S}_{\mu\nu}(\tilde{g}, \partial\tilde{g}, \partial^2\tilde{g}) - H(\tilde{g}, \chi, \pi), \tag{8}$$

where the derivatives $\partial L/\partial \tilde{R}$ and $\partial L/\partial \tilde{S}_{\mu\nu}$ are set equal to the canonical momenta $\chi + 1$ and $\pi^{\mu\nu}$ respectively, while the "velocities" $\tilde{R}$ and $\tilde{S}_{\mu\nu}$ explicitly depend on first

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and second derivatives of $\tilde{g}_{\mu\nu}$. In classical mechanics for a first order Lagrangian $L(q,\dot{q})$ one has

$$L_H(q,p,\dot{q},\dot{p}) \equiv p\dot{q} - H(q,p) = p\dot{q} - p\dot{q}(q,p) - L(q,\dot{q}(q,p)), \qquad (9)$$

i.e. L_H is a scalar function on the tangent bundle to the cotangent bundle to the configuration space; L_H does not depend on $\dot{p}$. Similarly, in a field theory L_H is independent of partial derivatives of canonical momenta. In classical mechnics the action $\int L_H \mathrm{d}t$ gives rise, when varied with respect to p, to the equation $\dot{q} = \partial H/\partial p$, while varied with respect to q generates

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_H}{\partial \dot{q}}\right) - \frac{\partial L_H}{\partial q} = 0; \qquad (10)$$

the latter equation is equivalent to

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \qquad (11)$$

Thus L_H simultaneously generates both Hamilton and Lagrange equations of motion. In the case of NLG theories one is interested in replacing the fourth order Lagrange equations by the equivalent second order Hamilton ones. For the Lagrangian (1) L_H reads

$$L_H = \tilde{R} + \chi\tilde{R} + \pi^{\mu\nu}\tilde{S}_{\mu\nu} - \frac{d}{4(ad+b)}\chi^2 - \frac{1}{4b}\pi^{\mu\nu}\pi_{\mu\nu}. \qquad (12)$$

One sees that the linear Hilbert–Einstein Lagrangian for the metric field is recovered. This means that Hamilton equations for $\tilde{g}_{\mu\nu}$ are not just second order ones of any kind but exactly Einstein field equations[3] $\tilde{G}_{\mu\nu} = \tilde{T}_{\mu\nu}(\tilde{g},\chi,\pi)$. The nonminimal coupling interaction terms $\chi\tilde{R}$ and $\pi^{\mu\nu}\tilde{S}_{\mu\nu}$ will cause that $\tilde{T}_{\mu\nu}$ will depend on second derivatives of χ and $\pi^{\mu\nu}$ and will contain Ricci tensor. Since Hilbert–Einstein Lagrangian for the metric field in general relativity is $L_g(\tilde{g}) = \frac{1}{2}\tilde{R}$, then $L_H = 2L_g + 2L_f$, where L_f is the Lagrangian for the non–geometric components of gravity, χ and $\pi^{\mu\nu}$. The energy–momentum tensor is then

$$-\frac{1}{2}\sqrt{-\tilde{g}}\,\tilde{T}_{\mu\nu}(\tilde{g},\chi,\pi) \equiv \frac{\delta}{\delta\tilde{g}^{\mu\nu}}(\sqrt{-\tilde{g}}\,L_f) \qquad (13)$$

$$= \frac{1}{2}\frac{\delta}{\delta\tilde{g}^{\mu\nu}}\left[\sqrt{-\tilde{g}}\left(\chi\tilde{R} + \pi^{\alpha\beta}\tilde{S}_{\alpha\beta} - \frac{d}{4(ad+b)}\chi^2 - \frac{1}{4b}\pi^{\alpha\beta}\pi_{\alpha\beta}\right)\right].$$

Explicitly the equations $\frac{\delta}{\delta\tilde{g}^{\mu\nu}}L_H = 0$ read

$$\tilde{G}_{\mu\nu}(\tilde{g}) = \tilde{T}_{\mu\nu}(\tilde{g},\chi,\pi) = -\chi\tilde{G}_{\mu\nu} + \chi_{;\mu\nu} - \tilde{g}_{\mu\nu}\,\tilde{\Box}\chi + \frac{1}{2}\tilde{R}_{\alpha\beta}\pi^{\alpha\beta}\tilde{g}_{\mu\nu} -$$

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[3]We use units $8\pi G = c = 1$, the signature is $(-+++)$. We use all the conventions of [22].

$$-\frac{1}{2}\pi^{\alpha\beta}{}_{;\alpha\beta}\tilde{g}_{\mu\nu} + \pi^{\alpha}{}_{(\mu;\nu)\alpha} - \frac{1}{2}\pi_{\mu\nu;\alpha}{}^{;\alpha} - \frac{d}{8(ad+b)}\chi^2\tilde{g}_{\mu\nu} -$$

$$-\frac{1}{8b}\pi^{\alpha\beta}\pi_{\alpha\beta}\tilde{g}_{\mu\nu} - \frac{1}{2b}\pi_{\mu}{}^{\alpha}\pi_{\alpha\nu} - \frac{1}{d}\tilde{R}\pi_{\mu\nu}, \tag{14}$$

here $f_{;\alpha}$ denotes the covariant derivative with respect to $\tilde{g}_{\mu\nu}$ and $\tilde{\Box}f = \tilde{g}^{\mu\nu}f_{;\mu\nu} = f_{;\mu}{}^{;\mu}$. As remarked above $\tilde{T}_{\mu\nu}$ more resembles the stress tensor for the conformally invariant scalar field [23, 24] than that for ordinary matter. The equations of motion for χ and $\pi^{\mu\nu}$ are purely algebraic and clearly coincide with (5),

$$\frac{\delta L_H}{\delta\chi} = 0 \Rightarrow \tilde{R} = \frac{d}{2(ad+b)}\chi, \qquad \frac{\delta L_H}{\delta\pi^{\mu\nu}} = 0 \Rightarrow \tilde{S}_{\mu\nu} = \frac{1}{2b}\pi_{\mu\nu}. \tag{15}$$

These equations can be recast in the form of Einstein ones,

$$\tilde{G}_{\mu\nu}(\tilde{g}) = \frac{1}{2b}\pi_{\mu\nu} - \frac{d-2}{4(ad+b)}\chi\tilde{g}_{\mu\nu}. \tag{16}$$

Comparison of eqs. (14) and (16) shows that *for solutions* there exists a simple linear expression for the stress tensor:

$$\tilde{T}_{\mu\nu}(\tilde{g}, \chi, \pi) = \frac{1}{2b}\pi_{\mu\nu} - \frac{d-2}{4(ad+b)}\chi\tilde{g}_{\mu\nu}. \tag{17}$$

This relationship allows one to derive differential propagation equations for χ and $\pi^{\mu\nu}$. Before doing it we simplify the expression (14) for $\tilde{T}_{\mu\nu}$ with the aid of (16) by replacing $\tilde{R}_{\mu\nu}$ by χ and $\pi^{\mu\nu}$ and making use of the Bianchi identity for $\tilde{G}_{\mu\nu}$. The latter provides a first order constraint on χ and $\pi^{\mu\nu}$,

$$\pi^{\mu\nu}{}_{;\nu} = \frac{(d-2)b}{2(ad+b)}\chi^{,\mu}, \tag{18}$$

the constraint is already solved with respect to $\chi^{,\mu}$. Upon inserting (16) into the r.h.s. of (14) one gets

$$\tilde{T}_{\mu\nu}(\tilde{g}, \chi, \pi) = \chi_{;\mu\nu} - \frac{1}{2}\tilde{\Box}\pi_{\mu\nu} + \pi^{\alpha}{}_{(\mu;\nu)\alpha} - \frac{1}{2b}\pi_{\mu}{}^{\alpha}\pi_{\alpha\nu} - \frac{ad+2b}{2(ad+b)b}\chi\pi_{\mu\nu}$$

$$+ \tilde{g}_{\mu\nu}\left(-\tilde{\Box}\chi - \frac{1}{2}\pi^{\alpha\beta}{}_{;\alpha\beta} + \frac{d-4}{8(ad+b)}\chi^2 + \frac{1}{8b}\pi^{\alpha\beta}\pi_{\alpha\beta}\right). \tag{19}$$

Equating the trace $\tilde{T}_{\mu\nu}\tilde{g}^{\mu\nu}$ computed from (17) to the trace of (19) and applying (18) one arrives at a quasilinear equation of motion for χ,

$$[4(d-1)a + db]\,\tilde{\Box}\chi - \frac{d-4}{2d}\left(\frac{ad+b}{b}\pi^{\alpha\beta}\pi_{\alpha\beta} + d\chi^2\right) - (d-2)\chi = 0. \tag{20}$$

The field χ is self–interacting and is coupled to $\pi^{\mu\nu}$, the term $\pi^{\alpha\beta}\pi_{\alpha\beta}$ acts as a source for χ. Eq. (20) is a Klein–Gordon equation with a potential and an external source. The mass of χ is

$$m_\chi^2 = \frac{d-2}{4(d-1)a + db} \tag{21}$$

and may be of both signs depending on the parameters.

To derive a propagation equation for $\pi^{\mu\nu}$ one replaces derivatives of χ in (19) by derivatives of $\pi^{\mu\nu}$ with the help of (18). Then $\tilde{T}_{\mu\nu}$ depends on χ via terms $\chi\pi_{\mu\nu}$ and $\chi^2\tilde{g}_{\mu\nu}$; the latter is eliminated with the aid of eq. (20) and the reappearing term $\tilde{\Box}\chi$ is again removed employing (18). The resulting expression for $\tilde{T}_{\mu\nu}$, which contains terms $\chi\pi_{\mu\nu}$ and $\chi\tilde{g}_{\mu\nu}$, is set equal to the r.h.s. of eq. (17), then the terms $\chi\tilde{g}_{\mu\nu}$ cancel each other and finally one arrives at the following equation of motion for $\pi_{\mu\nu}$,

$$\tilde{\Box}\pi_{\mu\nu} - \frac{4(ad+b)}{(d-2)b}\pi_{\alpha(\mu}{}^{;\alpha}{}_{;\nu)} - 2\pi^\alpha{}_{(\mu;\nu)\alpha} + \frac{1}{b}\pi_{\mu\nu} + \frac{1}{b}\pi_{\mu\alpha}\pi_\nu{}^\alpha +$$
$$+ \frac{ad+2b}{(ad+b)b}\chi\pi_{\mu\nu} + \tilde{g}_{\mu\nu}\left(\frac{2(2a+b)}{(d-2)b}\pi^{\alpha\beta}{}_{;\alpha\beta} - \frac{1}{bd}\pi^{\alpha\beta}\pi_{\alpha\beta}\right) = 0. \tag{22}$$

The triplet of gravitational fields is described by a coupled system of eqs. (16), (20) and (22) and the constraint (18). The equations (20) and (22) are quasilinear and contain interaction and self–interaction terms which cannot be removed for dimensions $d > 4$.

Since for generic dimension the dynamics of the fields $\pi_{\mu\nu}$ and χ cannot be decoupled, one can obtain some information on the individual behaviour of each field by considering particular solutions in which only one of the two fields is excited.

1. Let $\pi_{\mu\nu} = 0$. Then eq. (22) holds identically while the constraint (18) implies $\chi = \mathrm{const}$ and eq. (20) reduces to a quadratic equation, $(d-4)\chi^2 + 2(d-2)\chi = 0$. One solves it separately for $d > 4$ and for $d = 4$.

   a) $d > 4$. There are two solutions, $\chi = 0$ and $\chi = -2(d-2)/(d-4)$. From eq. (16) they correspond to $\tilde{R}_{\mu\nu} = 0$ and $\tilde{R}_{\mu\nu} = -\frac{d-2}{(d-4)(ad+b)}\tilde{g}_{\mu\nu}$ i.e. Einstein space respectively. Therefore there exist two distinct ground state solutions: Minkowski space for $\chi = 0$ and d–dimensional de Sitter space or anti–de Sitter for $\chi < 0$ depending on the sign of $ad + b$.

   b) For $d = 4$ the equation has only one solution $\chi = 0$ corresponding to $\tilde{R}_{\mu\nu} = 0$ and there is a unique ground state solution $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ (Minkowski metric) and $\pi_{\mu\nu} = \chi = 0$.

2. Let $\chi = 0$. We consider solutions for $d > 4$. The scalar field equation (20) generates the algebraic equation $\pi^{\alpha\beta}\pi_{\alpha\beta} = 0$, besides $\pi^{\mu\nu}{}_{;\nu} = 0$ arising from (18). Eq. (22) gets reduced, upon employing the two equations, to a linear equation

$$\tilde{\Box}\pi_{\mu\nu} + \frac{1}{b}\pi_{\mu\nu} - 2\tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} = 0. \tag{23}$$

This shows that the mass of the tensor field is $m_\pi^2 = -1/b$.

2.1 The four–dimensional case

Dimension four is clearly distinguished by the scalar field eq. (20). Setting $d = 4$, one finds two relevant properties:

(i) both the self–interaction and the source in eq. (20) vanish and χ satisfies the linear Klein–Gordon equation

$$2(3a + b)\,\tilde{\Box}\chi - \chi = 0; \tag{24}$$

for $3a + b > 0$ the mass is real,

$$m_\chi^2 = \frac{1}{2(3a + b)} > 0. \tag{25}$$

We shall see below that this allows one to decouple completely the propagation equations for the fields $\pi_{\mu\nu}$ and χ;

(ii) furthermore, for $3a + b = 0$ eq. (24) admits only one solution, $\chi = 0$, so for this particular choice of coefficients the scalar field disappears from the theory: we shall exploit this fact in the next section to concentrate our investigation only on the spin–two field.

We stress that for $d > 4$ vanishing of the coefficient of $\tilde{\Box}\chi$ in eq. (20) does not imply that that the scalar drops out. In fact, the equation gives rise to an algebraic relationship between the scalar and tensor fields,

$$\pi^{\alpha\beta}\pi_{\alpha\beta} = \frac{4d(d - 1)}{(d - 2)^2}\chi^2 + \frac{8d(d - 1)}{(d - 2)(d - 4)}\chi, \tag{26}$$

which replaces a propagation equation for χ.

To prove the statements above, one starts from eq. (22), and for $d = 4$ one eliminates χ from the interaction term applying eq. (24): then $\tilde{\Box}\chi$ is replaced by $\pi^{\alpha\beta}{}_{;\alpha\beta}$ with the help of (18). Then the equations of motion for $\pi_{\mu\nu}$ read

$$\tilde{\Box}\pi_{\mu\nu} - \frac{2}{b}(4a + b)\pi_{\alpha(\mu}{}^{;\alpha}{}_{;\nu)} - 2\pi^\alpha{}_{(\mu;\nu)\alpha} + \frac{1}{b}\pi_{\mu\nu} + \frac{1}{b}\pi_{\mu\alpha}\pi_\nu{}^\alpha +$$

$$+ \frac{4}{b^2}(2a + b)(3a + b)\pi^{\alpha\beta}{}_{;\alpha\beta}\pi_{\mu\nu} + \tilde{g}_{\mu\nu}\left(\frac{2a + b}{b}\pi^{\alpha\beta}{}_{;\alpha\beta} - \frac{1}{4b}\pi^{\alpha\beta}\pi_{\alpha\beta}\right) = 0. \tag{27}$$

The linear equation (24) for χ and quasilinear eq. (27) for $\pi_{\mu\nu}$ are decoupled. Equations (27) are linearly dependent since the trace (w.r.t. $\tilde{g}^{\mu\nu}$) vanishes identically, hence there are 9 algebraically independent equations (they satisfy also differential identities, see below).

For the special solution $\chi = 0$ the differential constraint reads $\pi^{\mu\nu}{}_{;\nu} = 0$. Then eq. (27) reduces to

$$\tilde{\Box}\pi_{\mu\nu} + \frac{1}{b}\pi_{\mu\nu} - 2\tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} - \frac{1}{4b}\pi^{\alpha\beta}\pi_{\alpha\beta}\tilde{g}_{\mu\nu} = 0 \tag{28}$$

and the mass of the field is the same as in $d > 4$, i.e. $m_\pi^2 = -1/b$. The masses of the massive components of the gravitational triplet, $m_\chi^2 = [2(3a + b)]^{-1}$ and $m_\pi^2 = -b^{-1}$, agree with the values found in the linear approximation to the quadratic Lagrangian for spin–0 and spin–2 fields by [16, 25] and [26]. The non–tachyon condition is then $b < 0$ and $3a + b > 0$ [7].

This condition shows that the $\tilde{R}^2$ term in the Lagrangian (1) is essential. In fact, if this term is absent and the Lagrangian reads $L = \tilde{R} + b\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}$, then (for dimensionality four)

$$\tilde{G}_{\mu\nu} = \tilde{T}_{\mu\nu}(\tilde{g}, \chi, \pi) = \frac{1}{2b}(\pi_{\mu\nu} - \chi\tilde{g}_{\mu\nu}), \tag{29}$$

$$\tilde{\Box}\chi - \frac{1}{2b}\chi = 0 \tag{30}$$

and

$$\tilde{\Box}\pi_{\mu\nu} + \frac{1}{b}(1 + \chi)\pi_{\mu\nu} - 2\tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} + \frac{1}{2b}\tilde{g}_{\mu\nu}\left(\chi - \frac{1}{2}\pi^{\alpha\beta}\pi_{\alpha\beta}\right) - 4\chi_{;\mu\nu} = 0. \tag{31}$$

The two fields have masses $m_\chi^2 = 1/(2b)$ and $m_\pi^2 = -1/b$ and one of them is necessarily a tachyon. It is worth noting that in the case of restricted NLG theories, i.e. $L = f(\tilde{R})$, the $a\tilde{R}^2$ term is also essential in the Taylor expansion of the Lagrangian: for $a > 0$ Minkowski space is stable as the ground state solution of the theory, while for $a < 0$ it is classically unstable and for $a = 0$ the solution may be unstable or stable [15].

Finally we show that all nine algebraically independent equations (27) are hyperbolic propagation equations for $\pi_{\mu\nu}$ and they contain no constraint equations. To this end one replaces $\pi^{\mu\nu}{}_{;\nu}$ by $\chi^{,\mu}$ with the aid of (18) and one arrives at the following equations:

$$\tilde{\Box}\pi_{\mu\nu} + \frac{1}{b}\pi_{\mu\nu} + \frac{1}{b}\pi_{\mu\alpha}\pi_\nu{}^\alpha - \frac{1}{4b}\tilde{g}_{\mu\nu}\pi^{\alpha\beta}\pi_{\alpha\beta} - 2\tilde{R}_{\alpha(\mu}\pi_{\nu)}{}^\alpha - 2\tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} =$$
$$= \frac{2a + b}{4a + b}\left[4\chi_{;\mu\nu} - \frac{1}{2(3a + b)}\chi\tilde{g}_{\mu\nu} - \frac{2}{b}\chi\pi_{\mu\nu}\right]. \tag{32}$$

In this form the hyperbolicity of all the equations is evident.

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3 The spin–2 field in the gravitational doublet in Helmholtz–Jordan frame

We have seen in the previous section that the generic quadratic Lagrangian (1), subject only to the non–tachyon condition $b < 0$ and $3a + b > 0$, describes a triplet of gravitational fields, HJF= $\{\tilde{g}_{\mu\nu}, \chi, \pi^{\mu\nu}\}$, where the nongeometric fields in the triplet represent (on the quantum–mechanical level) interacting particles with positive masses. It has long been known that the theory (1) has 8 degrees of freedom [25, 27, 28, 29]. As it was first found in the linear approximation [16] and then in the exact theory [13, 12], these degrees of freedom are carried by a massless spin–2 field (graviton, 2 degrees of freedom), a massive spin–2 field (5 d.o.f.) and a massive scalar field.

We are interested in the dynamics and physical properties of the massive spin–2 field. In this context, the scalar is undesirable and its presence only makes the system of the equations of motion more involved. One can get rid of the unwanted scalar by a proper choice of the coefficients in the original Lagrangian. As mentioned previously, for $3a + b = 0$ eq. (24) has only one trivial solution[4] $\chi = 0$. We therefore restrict our further study to the special Lagrangian

$$L = \tilde{R} + a(\tilde{R}^2 - 3\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}) \tag{33}$$

and denote $m^2 \equiv (3a)^{-1}$ assuming $a > 0$. As it was noticed in [16, 13] this Lagrangian can be neatly expressed in terms of Weyl tensor,

$$L = \tilde{R} + \frac{1}{2m^2}(L_{GB} - \tilde{C}_{\alpha\beta\mu\nu}\tilde{C}^{\alpha\beta\mu\nu}) \tag{34}$$

where $L_{\mathrm{GB}} = \tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\alpha\beta\mu\nu} - 4\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu} + \tilde{R}^2$, the Gauss–Bonnet term, is a total divergence in four dimensions.

One formally repeats the operations of the previous section and replaces the expressions (15) to (19) by

$$\tilde{R} = 6m^2\chi, \qquad \tilde{S}_{\mu\nu} = -\frac{m^2}{2}\pi_{\mu\nu}, \tag{35}$$

$$\tilde{G}_{\mu\nu} = \tilde{T}_{\mu\nu}(\tilde{g}, \chi, \pi) = -\frac{m^2}{2}(\pi_{\mu\nu} + 3\chi\tilde{g}_{\mu\nu}), \tag{36}$$

$\pi^{\mu\nu}{}_{;\nu} = -3\chi^{,\mu}$ and

$$\tilde{T}_{\mu\nu}(\tilde{g}, \chi, \pi) = \chi_{;\mu\nu} - \frac{1}{2}\tilde{\Box}\pi_{\mu\nu} + \pi^{\alpha}{}_{(\mu;\nu)\alpha} + \frac{m^2}{2}\pi_{\mu}{}^{\alpha}\pi_{\alpha\nu} - m^2\chi\pi_{\mu\nu} - $$
$$-\tilde{g}_{\mu\nu}\left(\tilde{\Box}\chi + \frac{1}{2}\pi^{\alpha\beta}{}_{;\alpha\beta} + \frac{m^2}{8}\pi^{\alpha\beta}\pi_{\alpha\beta}\right) \tag{37}$$

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[4] That the scalar degree of freedom disappears in this case was previously found in [16, 12].

respectively. The scalar field is still present in these equations. However the trace of eq. (37) is $\tilde{T}_{\mu\nu}\tilde{g}^{\mu\nu} = 0$ while the trace of eq. (36) yields $\tilde{T}_{\mu\nu}\tilde{g}^{\mu\nu} = -6m^2\chi$. Then $\chi = 0$ and the scalar drops out from the theory. Although the scalar field vanishes one cannot, however, remove it from the Lagrangian in HJF unless one imposes the constraint $\tilde{R} = 0$ already on the level of the initial Lagrangian in HF. It is more convenient to deal with Lagrangians containing no Lagrange multipliers and therefore the auxiliary non-dynamic (i.e. having no physical degrees of freedom) scalar χ remains in the Helmholtz Lagrangian (12) which now reads

$$L_H = \tilde{R} + \chi\tilde{R} + \pi^{\mu\nu}\tilde{S}_{\mu\nu} - 3m^2\chi^2 + \frac{m^2}{4}\pi^{\mu\nu}\pi_{\mu\nu}. \tag{38}$$

The system of field equations for the gravitational doublet HJF=$\{\tilde{g}_{\mu\nu}, \pi^{\mu\nu}\}$, having together seven degrees of freedom, consists of Einstein field equations,

$$\tilde{G}_{\mu\nu}(\tilde{g}) = \tilde{T}_{\mu\nu}(\tilde{g}, \pi) = -\frac{m^2}{2}\pi_{\mu\nu} \tag{39}$$

and quasilinear hyperbolic propagation equations for $\pi_{\mu\nu}$,

$$\tilde{\Box}\pi_{\mu\nu} - m^2\pi_{\mu\nu} - 2\tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} + \frac{m^2}{4}\tilde{g}_{\mu\nu}\pi^{\alpha\beta}\pi_{\alpha\beta} = -m^2\pi_{\mu\nu} - 2\tilde{T}_{\mu\nu}(\tilde{g}, \pi) = 0. \tag{40}$$

It should be stressed that, as is clearly seen from the method of deriving eqs. (20) and (22), the eqs. (39) and (40) are not simply the variational equations $\delta L_H/\delta\tilde{g}^{\mu\nu} = 0$ and $\delta L_H/\delta\pi^{\mu\nu} = 0$ with the substitutions $\chi = 0$ and $\tilde{R} = 0$.

The field $\pi^{\mu\nu}$ satisfies 5 constraints, $\pi^{\mu\nu}\tilde{g}_{\mu\nu} = 0$ and $\pi^{\mu\nu}_{;\nu} = 0$. Notice that the field equations give rise to no further constraints. In fact, the trace of (40) and divergence of (39) vanish identically if the constraints hold. A possible source of a further constraint is divergence of eq. (40). It may be shown by a direct calculation that if the equations (39) and (40) hold throughout the spacetime and if the five constraints are satisfied everywhere, then divergence of eq. (40) vanishes identically. This confirms the previous result [16, 25, 26, 12, 13, 7] that this field has spin two without any admixture of lower spin fields.

We now investigate the internal consistency of the theory based on eqs. (39) and (40). It is well known [17, 30, 29] that a linear spin–two field $\Phi_{\mu\nu}$, massive or massless, has inconsistent dynamics in the presence of gravitation since in a curved non-empty spacetime the field loses the degrees of freedom it had in flat spacetime and the five conditions which ensured there its purely spin–2 character, $\Phi_{\mu\nu}\eta^{\mu\nu} = 0 = \Phi^{\mu\nu}_{,\nu}$, are replaced by four differential constraints imposed on the field and Ricci tensor[5]. Here

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[5]For linear fields with spins higher than 2 it was long believed [31] that there was no easy way to have physical fields on anything but Minkowski space; only recently a progress has been made for massive fields [32].

we are dealing with the nonlinear spin–2 field $\pi_{\mu\nu}$ and one expects that this field is consistent. This expectation follows from the dynamical equivalence of the Helmholtz Lagrangian (38) to the purely metric Lagrangian (33) of the fourth–order theory and the latter one (as well as any other NLG theory with a Lagrangian being any smooth scalar function of the curvature tensor) is known to be consistent. However the spin–2 field theory in HJF should be a self-contained one and one should show its consistency without invoking its equivalent fourth–order version.

We first notice a difference in the structures of the theories for the linear and nonlinear spin–two fields. For the linear field $\Phi_{\mu\nu}$ in Minkowski space one first derives (quite involved) Lagrange field equations and then either by employing the gauge invariance (for the massless field) or by taking the trace and divergence of the field equations for the massive one, one derives the five constraints $\Phi_{\mu\nu}\eta^{\mu\nu} = 0 = \Phi^{\mu\nu}{}_{,\nu}$. The method does not work in a generic curved spacetime [17]. For the nonlinear field $\pi_{\mu\nu}$ the tracelessness is a direct consequence of the tracelessness of $\tilde{S}_{\mu\nu}$ and of the field eq. (15), which now reads (in terms of $L_f = \frac{1}{2}(L_H - \tilde{R})$ rather than of L_H)

$$\frac{\delta L_f}{\delta \pi^{\mu\nu}} \equiv E_{\mu\nu}(\pi) = \frac{1}{2}\tilde{S}_{\mu\nu} + \frac{m^2}{4}\pi_{\mu\nu} = 0; \tag{41}$$

in the Helmholtz Lagrangian (38) it is not assumed that $\pi_{\mu\nu}$ has vanishing trace. After eliminating the scalar field one gets $\tilde{R} = 0$ and the other four constraints $\pi^{\mu\nu}{}_{;\nu} = 0$ and the algebraic equation for $\pi_{\mu\nu}$ reduces to

$$E_{\mu\nu} = \frac{1}{2}(\tilde{R}_{\mu\nu} + \frac{m^2}{2}\pi_{\mu\nu}) = 0, \tag{42}$$

while the Einstein field equations for $\tilde{g}_{\mu\nu}$ are

$$\frac{\delta}{\delta \tilde{g}^{\mu\nu}}\left(\frac{1}{2}\sqrt{-\tilde{g}}L_H\right) = \tilde{G}_{\mu\nu} - \tilde{T}_{\mu\nu}(\tilde{g}, \pi) = 0, \tag{43}$$

where $\tilde{T}_{\mu\nu}$ is given by eq. (37) for $\chi = 0$. Hence the constraints ensuring that $\pi_{\mu\nu}$ has 5 degrees of freedom hold whenever the field equations hold. Clearly the system (42)–(43) is equivalent to the system (39)–(40) but the former is more convenient for studying the consistency of the equations. To this end one employs the coordinate invariance of the action integral [30]. Under an infinitesimal coordinate transformation $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$, $|\varepsilon^{\mu}| \ll 1$, the metric and the spin-2 field vary as $\delta\tilde{g}^{\mu\nu} = 2\varepsilon^{(\mu;\nu)}$ and

$$\delta\pi^{\mu\nu} \equiv -L_{\varepsilon}\pi^{\mu\nu} = -\pi^{\mu\nu}{}_{;\alpha}\varepsilon^{\alpha} + \pi^{\alpha\nu}\varepsilon^{\mu}_{;\alpha} + \pi^{\mu\alpha}\varepsilon^{\nu}_{;\alpha}, \tag{44}$$

here L is the Lie derivative. The action integral

$$S_f = \int L_f \sqrt{-\tilde{g}} \, \mathrm{d}^4x \tag{45}$$

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is generally covariant, therefore it is invariant under the transformation

$$\delta_\varepsilon S_f = \int \varepsilon^\mu \left[\tilde{T}_{\mu\nu}{}^{;\nu} - F\chi_{,\mu} - E_{\alpha\beta}\pi^{\alpha\beta}{}_{;\mu} - 2E_{\mu\alpha;\beta}\pi^{\alpha\beta} - 2E_{\mu\beta}\pi^{\alpha\beta}{}_{;\alpha} \right] \sqrt{-\tilde{g}} \; \mathrm{d}^4 x = 0, \quad (46)$$

where $F \equiv \delta L_f/\delta\chi = \frac{1}{2}\tilde{R} - 3m^2\chi$ and $\delta\chi = -\varepsilon^\mu\chi_{,\mu}$. Thus the coordinate invariance implies a strong Noether conservation law

$$\tilde{T}_{\mu\nu}{}^{;\nu} - F\chi_{,\mu} - E_{\alpha\beta}\pi^{\alpha\beta}{}_{;\mu} - 2E_{\mu\alpha;\beta}\pi^{\alpha\beta} - 2E_{\mu\beta}\pi^{\alpha\beta}{}_{;\alpha} = 0. \qquad (47)$$

Now assume that the field equations $F = 0$ and $E_{\mu\nu} = 0$ hold. Then also $E_{\mu\alpha;\beta} = 0$ holds and the identity reduces to $\tilde{T}_{\mu\nu}{}^{;\nu} = 0$. This is a consistency condition for Einstein field equations (43). Divergence of $\tilde{T}_{\mu\nu}$ should vanish due to the system of field equations and constraints without giving rise to further independent constraints.
By a direct calculation one proves the following proposition: if the field equations $\chi = 0$ and (42) and the five constraints $\pi^{\mu\nu}\tilde{g}_{\mu\nu} = 0 = \pi^{\mu\nu}{}_{;\nu}$ hold throughout the spacetime, then the stress tensor given by eq. (37) is divergenceless, $\tilde{T}_{\mu\nu}{}^{;\nu} \equiv 0$. This shows that the system (42)–(43) is consistent.

Owing to the tracelessness of $\pi^{\mu\nu}$ there is only one ground state solution for the system (39)–(40) [13, 18] (i.e. the spacetime is maximally symmetric and $\pi^{\mu\nu}$ is covariantly constant). This is Minkowski space, $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\pi_{\mu\nu} = 0$. This state is linearly stable since small metric perturbations $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and excitations of $\pi_{\mu\nu}$ around $\pi_{\mu\nu} = 0$ can be expanded into plane waves $h_{\mu\nu} = \pi_{\mu\nu} = p_{\mu\nu}\exp(ik_\alpha x^\alpha)$ with a constant wave vector k_α, $k_\alpha k^\alpha = -m^2$ and a constant wave amplitude $p_{\mu\nu}$ satisfying $p_{\mu\nu}\eta^{\mu\nu} = 0 = p_{\mu\nu}k^\nu$.

4 Massless spin–two field in HJF

The massive spin–2 field $\pi_{\mu\nu}$ has a finite range with the length scale m^{-1}. According to the principle of physical continuity [29] as the mass tends to zero, the long-range force mediated by $\pi_{\mu\nu}$ should have a smooth limit and in this limit it should coincide with the strictly infinite-range theory. We therefore consider two cases: first the exactly massless theory resulting from the Helmholtz Lagrangian for $m = 0$ and then the field equations of the previous section in the limit $m \to 0$.

After setting $m = 0$ in eq. (38) it is convenient to express the resulting Lagrangian in terms of Einstein tensor $\tilde{G}_{\mu\nu} = \tilde{S}_{\mu\nu} - \frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu}$ and the trace of the spin–2 field, $\pi = \pi^{\mu\nu}\tilde{g}_{\mu\nu}$. One gets

$$L_H = \tilde{R} + (\chi + \frac{1}{4}\pi)\tilde{R} + \pi^{\mu\nu}\tilde{G}_{\mu\nu}. \qquad (48)$$

Owing to the Bianchi identity this Lagrangian is invariant under the gauge transformation

$$\pi^{\mu\nu} \to \pi'^{\mu\nu} = \pi^{\mu\nu} + \varepsilon^{\mu;\nu} + \varepsilon^{\nu;\mu} \qquad \text{and} \qquad \chi \to \chi' = \chi - \frac{1}{2}\varepsilon^{\alpha}_{;\alpha} \tag{49}$$

with arbitrary vector ε^{μ}[6]. The scalar $\chi + \frac{1}{4}\pi$ is gauge invariant. We notice that the scalar χ cannot be removed already on the level of the Lagrangian L_H since it would break the gauge invariance. Moreover the term $\chi\tilde{R}$ in L_H is essential to obtain appropriate field equations. Hence in this approach the scalar cannot be eliminated by using the first principles. One can only set $\chi = 0$ in a specific gauge.

The field equations are

$$\frac{\delta L_H}{\delta \chi} = \tilde{R} = 0 \qquad \text{and} \qquad \frac{\delta L_H}{\delta \pi^{\mu\nu}} = \tilde{R}_{\mu\nu} - \frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu} = 0,$$

which imply $\tilde{R}_{\mu\nu} = 0$. It is clear that unlike in the massive case now one cannot recover the original fourth-order Lagrangian (1) since the two fields are here unrelated to Ricci tensor. The fields χ and $\pi_{\mu\nu}$ do not interact with the metric $\tilde{g}_{\mu\nu}$ which acts as an empty–spacetime background metric. This suggests in turn that the two components of the gravitational triplet are test fields on the metric background, e.g. some excitations, and one may expect that they satisfy linear equations of motion. Variation of L_H with respect to the metric yields Einstein field equations which are reduced in comparison to eq. (14),

$$\tilde{G}_{\mu\nu} = \tilde{T}_{\mu\nu}(\tilde{g}, \chi, \pi) = \chi_{;\mu\nu} - \tilde{g}_{\mu\nu}\,\tilde{\Box}\chi - \frac{1}{2}\,\tilde{\Box}\pi_{\mu\nu} - \frac{1}{2}\pi^{\alpha\beta}_{\;\;;\alpha\beta}\tilde{g}_{\mu\nu} + \pi^{\alpha}_{\;(\mu;\nu)\alpha} = 0; \tag{50}$$

the last equality follows from $\tilde{R}_{\mu\nu} = 0$. The scalar field χ does not appear here in the combination $\chi + \frac{1}{4}\pi$ as in the Helmholtz Lagrangian since upon varying with respect to the metric $\tilde{g}^{\mu\nu}$ one has $\delta\chi = 0$ while $\delta\pi = \pi^{\mu\nu}\delta\tilde{g}_{\mu\nu}$ ($\pi^{\mu\nu}$ is a fundamental field, i.e. is independent of the metric).

The harmonic gauge condition $\pi^{\mu\nu}_{\;\;;\nu} = 0$ is most convenient and the field equations for χ and $\pi^{\mu\nu}$ simplify (upon employing $\tilde{R}_{\mu\nu} = 0$) to

$$\tilde{G}_{\mu\nu} = \chi_{;\mu\nu} - \tilde{g}_{\mu\nu}\,\tilde{\Box}\chi - \frac{1}{2}\,\tilde{\Box}\pi_{\mu\nu} + \tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} = 0; \tag{51}$$

Their trace provides an equation for χ,

$$\tilde{\Box}(6\chi + \pi) = 0. \tag{52}$$

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[6]Another possible decomposition of $\tilde{g}_{\mu\nu}$ in this frame into fields with definite spin, as is done in [13], yields a different gauge transformation not affecting the scalar field.

The harmonic gauge condition does not fix the gauge uniquely and the remaining gauge freedom is generated by any vector ε_μ satisfying the equation

$$\widetilde{\Box}\varepsilon_\mu + \varepsilon^\alpha{}_{;\alpha\mu} = 0. \tag{53}$$

The scalar function $6\chi + \pi$ is not gauge invariant and under the transformation (49) varies as $6\chi' + \pi' = 6\chi + \pi - \varepsilon^\alpha{}_{;\alpha}$. For the residual gauge freedom the scalar $\varepsilon^\alpha{}_{;\alpha}$ satisfies the equation $\widetilde{\Box}\varepsilon^\alpha{}_{;\alpha} = 0$ which follows from (53) upon taking its divergence. Since both $6\chi + \pi$ and $\varepsilon^\alpha{}_{;\alpha}$ are solutions to the scalar wave equation, one can choose ε^μ such that $\varepsilon^\alpha{}_{;\alpha} = 6\chi + \pi$. The residual gauge freedom allows then one to use a specific gauge wherein $6\chi + \pi = 0$. Replacing χ by π one gets equations of motion for $\pi_{\mu\nu}$,

$$\widetilde{\Box}\pi_{\mu\nu} - 2\tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} + \frac{1}{3}\pi_{;\mu\nu} - \frac{1}{3}\tilde{g}_{\mu\nu}\,\widetilde{\Box}\pi = 0. \tag{54}$$

The trace of these equations vanishes identically, hence there are 9 algebraically independent equations. The scalar π is still present in (54) and $\pi_{\mu\nu}$ is subject to four constraints, thus it represents 6 degrees of freedom.

In this gauge one can take a special solution where the nondynamic field χ is zero. Then $\pi = 0$ too and the field $\pi_{\mu\nu}$ is subject to

$$\widetilde{\Box}\pi_{\mu\nu} - 2\tilde{R}_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} = 0 \tag{55}$$

and the five constraints $\pi^{\mu\nu}{}_{;\nu} = 0 = \pi$, hence it has a definite spin equal two.
It may be shown that eqs. (55) do not generate further constraints since their divergence vanishes identically provided $\tilde{R}_{\mu\nu} = 0$ and $\pi^{\mu\nu}{}_{;\nu} = 0$.

Secondly, one takes the limit $m \to 0$ in the equations of motion (39) and (40). Since the scalar χ has already been eliminated, the resulting field equations, $\tilde{R}_{\mu\nu} = 0$ and (55), represent the full set of solutions rather than a special class; the five constraints hold.

In both cases the linear spin–2 field $\pi^{\mu\nu}$ is dynamically equivalent to gravitational perturbations of the background empty-spacetime metric, $\tilde{R}_{\mu\nu}(\tilde{g}) = 0$. If the perturbed metric is $\tilde{g}_{\mu\nu} + h_{\mu\nu}$, then on the level of the field equations and in the traceless–transversal gauge one can identify $h_{\mu\nu}$ with $\pi_{\mu\nu}$. In this sense the theory for a massless spin–2 field $\pi_{\mu\nu}$ resulting from the Lagrangian (33) is trivial. Yet [13] claim that from the form of their Lagrangian for the field, which is very similar to (48), one can infer that the field is ghostlike. Since both Lagrangians in HJF have no kinetic terms, this conclusion can be reliably derived only in Einstein frame and this will be done in Sect.7.

5 Equations of motion for the gravitational doublet in Einstein frame: a generic Lagrangian

We have seen that the multiplet of fields forming HJF, in which the unifying field $\tilde{g}_{\mu\nu}$ was decomposed, is suitable for investigating the dynamical evolution, consistency and particle content of the theory. This, however, does not imply that the fields of this frame are truly physical variables. The problem of physical variables first appeared for a system of a scalar field interacting with gravity, since in this system one can make arbitrary redefinitions (see [15] and references therein). Following [15] we say that different formulations (employing different frames) of a theory provide various versions of the same theory if the frames are dynamically equivalent. Dynamical equivalence means that Lagrange equations of motion in different frames are equivalent while their action integral are, in general, unrelated; in classical field theory this is sufficient to regard these frames as various manifestations of one theory. Recently is has been shown that the two most important frames, JF and EF, are equivalent not only in the above classical sense: in the quantum context the path integral for the Lagrangian (1) in JF is equal to the path integral in EF for Einstein gravity coupled to a massive spin–2 field and a massive scalar [14]. In both frames many physical quantities are the same, e.g. for black holes all the thermodynamical dynamical variables do not alter under a suitable Legendre map [33] and the Zeroth Law and the Second Law of black hole thermodynamics are proved in EF [34] (it is worth noticing that the proof works provided the coefficient of the R^2 term in the Lagrangian in JF is positive). It is not quite clear whether the latter proof and many other computations mentioned in [15] mean that EF is the physical frame or merely show that it is computationally advantageous. The physical frame is distinguished among all possible dynamically equivalent ones in a rather subtle way: its field variables are operationally measurable and field excitations above the stable ground state solution have positive energy density (or satisfy the dominant energy condition). In the case of a scalar field appearing in scalar–tensor gravity theories or arising in the restricted NLG theories this criterion uniquely points to the physical variables (frame): the physical spacetime metric is conformally related (by a degenerate Legendre transformation) to the metric field of Jordan frame in which the theory has been originally formulated [15], and coincides with the EF metric. We emphasize that energy density is very sensitive to field redefinitions and thus is a good indicator of which variables are physical. In this sense the energy–momentum tensor for the spin–2 field, eq. (39), being just proportional to $\pi_{\mu\nu}$, is unphysical[7]. In fact, in absence of any empirical evidence regarding energy density for spin–2 particles, one may invoke analogy with a scalar field.

Though a scalar field also has not been observed yet, it is generally accepted that

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[7]Also a different definition of a tensor $\pi_{\mu\nu}$, representing the spin–two field in HJF, given in [13], results in a linear energy–momentum tensor.

its energy–momentum tensor should be purely quadratic in the field derivatives and no linear terms may appear (in the kinetic part). For any long–range scalar field the presence of such linear terms would cause difficulties in determining its total energy. Given any test scalar field Φ in an empty spacetime $(R_{\mu\nu}(g) = 0)$ one can take the tensor $\theta_{\mu\nu} = \Phi_{;\mu\nu} - g_{\mu\nu}\Box\Phi$, which is trivially conserved and dominates in the effective energy–momentum tensor at large distances and hence affects the total energy [35]. The degenerate Legendre transformation to the Einstein frame in scalar–tensor gravity and restricted NLG theories removes all linear terms and provides a fully acceptable expression for the scalar field energy density.

We therefore study now the other, more sophisticated way of decomposing $\tilde{g}_{\mu\nu}$ into a multiplet of fields by means of a Legendre transformation. We regard this transformation as a transition to the physical frame. We shall see that this transformation does not guarantee that the spin–2 field has positive energy density. Maybe the criterion for physical variables should be relaxed in the case of this field. In any case the new variables seem to be more physical than those in HJF.
One generates the physical spacetime metric $g_{\mu\nu}$ from $\tilde{g}_{\mu\nu}$ while the latter is viewed as a non-geometric component of gravity. It is useful and instructive to take at the outset the fully generic Lagrangian $L = f(\tilde{g}_{\mu\nu}, \tilde{R}_{\alpha\beta})$ and only later to specify it to the form (33). According to [9] and [10] the true spacetime metric is defined as[8]

$$g^{\mu\nu} \equiv (-\tilde{g})^{-1/2} \left| \det\left(\frac{\partial f}{\partial \tilde{R}_{\alpha\beta}}\right) \right|^{-1/2} \frac{\partial f}{\partial \tilde{R}_{\mu\nu}} = \left|\frac{\tilde{g}}{g}\right|^{1/2} \frac{\partial f}{\partial \tilde{R}_{\mu\nu}}, \tag{56}$$

where $\tilde{g} = \det(\tilde{g}_{\mu\nu})$, $g = \det(g_{\mu\nu})$ and $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$, $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$. To view $g^{\mu\nu}$ as a spacetime metric one assumes that $\det(\partial f/\partial \tilde{R}_{\alpha\beta}) \neq 0$. From now on all tensor indices will be raised and lowered with the aid of this metric. At this point, to make the following equations more readable, we alter our notation and denote the original tensor field $\tilde{g}_{\mu\nu}$ by $\psi_{\mu\nu}$ and its inverse $\tilde{g}^{\mu\nu}$ by $\gamma^{\mu\nu}$. The Legendre transformation (56) is a map of the metric manifold $(M, \psi_{\mu\nu})$ to another one, $(M, g_{\mu\nu})$. We will not consider here the hard problem of whether the transformation is globally invertible[9]; we assume that the map is regular in some neighbourhood of a ground state solution. The fields $g_{\mu\nu}$ and $\psi_{\mu\nu}$ will be referred to as Einstein frame, EF $= \{g_{\mu\nu}, \psi_{\mu\nu}\}$. For the generic Lagrangian $\psi_{\mu\nu}$ is actually a mixture of fields carrying spin two and zero. Notice that for f as in (1),

$$g^{\mu\nu} = \left|\frac{\psi}{g}\right|^{1/2} [(1 + 2a\tilde{R})\gamma^{\mu\nu} + 2b\tilde{R}^{\mu\nu}], \tag{57}$$

[8]In [12], [13] and [14] the physical metric is constructed in a more involved and tricky way; the outcome is equivalent to the Legendre map (56).

[9]Some basic considerations of the problem can be found in [13].

hence for $\psi_{\mu\nu}$ being Lorentzian and close to Minkowski metric, $g_{\mu\nu}$ is also Lorentzian and close to flat metric (and thus invertible). This shows the importance of the linear term in (1). Furthermore, in the limit $m_\pi \to \infty$, i.e. $b \to 0$, the spin–2 field $\pi_{\mu\nu}$ becomes a non-dynamic one since it is determined in terms of χ and $\psi_{\mu\nu}$ by an algebraic equation,

$$(1+\chi)\pi_{\mu\nu} + \pi_{\mu\alpha}\pi_\nu{}^\alpha - \frac{1}{4}\psi_{\mu\nu}\pi^{\alpha\beta}\pi_{\alpha\beta} = 0. \tag{58}$$

($\pi_{\mu\nu} = 0$ in the linear approximation.) Then (57) is reduced to the conformal rescaling of the metric [36] (for more references cf. [15]) being the degenerate Legendre transformation [9, 10],

$$g_{\mu\nu} = \left(\lim_{b \to 0} \frac{\partial f}{\partial \tilde{R}}\right)\psi_{\mu\nu} = (1 + 2a\tilde{R})\psi_{\mu\nu}. \tag{59}$$

As mentioned above, for a restricted NLG theory, $L = f(\tilde{R})$, Einstein frame is the physical one.

For the time being we investigate the generic Lagrangian. We shall use a tensor being the difference of the two Christoffel connections, [9, 13],

$$Q^\alpha_{\mu\nu}(g, \psi) \equiv \tilde{\Gamma}^\alpha_{\mu\nu}(\psi) - \Gamma^\alpha_{\mu\nu}(g) = \frac{1}{2}\gamma^{\alpha\beta}\left(\psi_{\beta\mu;\nu} + \psi_{\beta\nu;\mu} - \psi_{\mu\nu;\beta}\right), \tag{60}$$

hereafter $\nabla_\alpha T \equiv T_{;\alpha}$ denotes the covariant derivative of any T with respect to the physical metric $g_{\mu\nu}$. For any two metric tensors (not necessarily related by a transformation) the following identity holds for their curvatures [9],

$$\begin{aligned} K^\alpha{}_{\beta\mu\nu}(Q) &\equiv \tilde{R}^\alpha{}_{\beta\mu\nu}(\psi) - R^\alpha{}_{\beta\mu\nu}(g) = \\ &= Q^\alpha_{\beta\nu;\mu} - Q^\alpha_{\beta\mu;\nu} + Q^\sigma_{\beta\nu}Q^\alpha_{\sigma\mu} - Q^\sigma_{\beta\mu}Q^\alpha_{\sigma\nu} = -K^\alpha{}_{\beta\nu\mu}. \end{aligned} \tag{61}$$

This "curvature difference tensor" for Q satisfies $K^\alpha{}_{\alpha\mu\nu}(\psi, g) = 0$ and generates a "Ricci difference tensor"

$$K_{\mu\nu}(Q) \equiv \tilde{R}_{\mu\nu}(\psi) - R_{\mu\nu}(g) = Q^\alpha_{\mu\nu;\alpha} - Q^\alpha_{\alpha\mu;\nu} + Q^\alpha_{\mu\nu}Q^\beta_{\alpha\beta} - Q^\alpha_{\mu\beta}Q^\beta_{\nu\alpha} = K_{\nu\mu} \tag{62}$$

since $Q^\alpha_{\alpha\mu;\nu} = Q^\alpha_{\alpha\nu;\mu}$, and a "curvature difference scalar"

$$g^{\mu\nu}K_{\mu\nu}(\psi, g) = \nabla_\alpha\left(g^{\mu\nu}Q^\alpha_{\mu\nu} - g^{\alpha\mu}Q^\beta_{\beta\mu}\right) + g^{\mu\nu}\left(Q^\alpha_{\mu\nu}Q^\beta_{\alpha\beta} - Q^\alpha_{\mu\beta}Q^\beta_{\nu\alpha}\right). \tag{63}$$

Second order field equations in EF are generated by a Helmholtz Lagrangian [9, 10]. First one inverts the relationship (56), i.e. solves it with respect to Ricci tensor, $\tilde{R}_{\mu\nu}(\psi_{\alpha\beta}, \partial\psi_{\alpha\beta}, \partial^2\psi_{\alpha\beta}) = r_{\mu\nu}(g^{\alpha\beta}, \psi_{\alpha\beta})$. The functions $r_{\mu\nu}$ do not contain derivatives of $g^{\alpha\beta}$ and $\psi_{\alpha\beta}$. A unique solution exists providing that the Hessian

$$\det\left(\frac{\partial^2 f}{\partial \tilde{R}_{\alpha\beta}\tilde{R}_{\mu\nu}}\right) \neq 0.$$

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This means that f must explicitly depend (at least quadratically) on $\tilde{R}_{\mu\nu}$ and not only on $\tilde{R}$. It is here that the assumption that the original Lagrangian is at most quadratic is of practical importance. Next one constructs the Hamiltonian density (it is more convenient to use at this point scalar densities than pure scalars)

$$H(g,\psi) \equiv g^{\mu\nu} r_{\mu\nu} \sqrt{-g} - f\left(\psi_{\mu\nu}, r_{\alpha\beta}(g,\psi)\right) \sqrt{-\psi} \qquad (64)$$

and then a Helmholtz Lagrangian density,

$$L_H \sqrt{-g} \equiv g^{\mu\nu} \tilde{R}_{\mu\nu}(\psi) \sqrt{-g} - H(g,\psi). \qquad (65)$$

From (62) one finds $\tilde{R}_{\mu\nu}(\psi) = R_{\mu\nu}(g) + K_{\mu\nu}(Q)$, then

$$L_H(g,\psi) = R(g) + g^{\mu\nu} K_{\mu\nu}(Q) - g^{\mu\nu} r_{\mu\nu}(g,\psi) + \left|\frac{\psi}{g}\right|^{1/2} f\left(\psi_{\mu\nu}, r_{\alpha\beta}(g,\psi)\right). \qquad (66)$$

It is remarkable [9, 10] that in EF the Hilbert–Einstein Lagrangian for the spacetime metric is recovered and the kinetic part of the Lagrangian for $\psi_{\mu\nu}$ is universal, i.e. is independent of the choice of the scalar function f. The only reminiscence of the original Lagrangian L in JF is contained in the potential part of L_H. It was far from being obvious that it is possible to define the physical metric in such a way that the gravitational part of L_H is exactly $R(g)$. In this sense Einstein general relativity is a universal Hamiltonian image (under the Legendre map) of any NLG theory. The total divergence appearing in eq. (63) may be discarded and the kinetic Lagrangian for $\psi_{\mu\nu}$ reads

$$K(Q) \equiv g^{\mu\nu}(Q^{\alpha}_{\mu\nu} Q^{\beta}_{\alpha\beta} - Q^{\alpha}_{\mu\beta} Q^{\beta}_{\nu\alpha}), \qquad (67)$$

what ensures that Lagrange equations of motion will be of second order. The proof that L_H in EF is dynamically equivalent to L in JF is given in [10] and [9]. As in HJF we write $L_H = 2L_g + 2L_M$ with

$$L_M = \frac{1}{2} K - \frac{1}{2} g^{\mu\nu} r_{\mu\nu} + \frac{1}{2} \left|\frac{\psi}{g}\right|^{1/2} f(\psi_{\alpha\beta}, r_{\alpha\beta}(g,\psi)) \qquad (68)$$

being the Lagrangian for $\psi_{\mu\nu}$. This Lagrangian was found in a different way in [13] and [14] for a different decomposition of the original metric $\psi_{\mu\nu}$.

Equations of motion for the metric are Einstein ones, $G_{\mu\nu}(g) = T_{\mu\nu}(g,\psi)$ where as usual

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} (\sqrt{-g} L_M).$$

However finding out $T_{\mu\nu}$ directly from L_M requires a very long computation. To compute the dependence of $T_{\mu\nu}$ on the dynamical variables for solutions one can use an alternative

derivation. The fundamental differential relation between the two metrics, $\tilde{R}_{\mu\nu}(\psi_{\alpha\beta}) = r_{\mu\nu}(g^{\alpha\beta}, \psi_{\alpha\beta})$, which represents the inverse Legendre map and is recovered in EF as one of the Euler–Lagrange equations, is inserted in eq.(62) giving rise to $R_{\mu\nu}(g) = r_{\mu\nu} - K_{\mu\nu}$. Next, one takes into account that for solutions the variational energy–momentum tensor (or stress tensor) $T_{\mu\nu}$ should be equal to the Einstein tensor,

$$T_{\mu\nu}(g, \psi) = G_{\mu\nu}(g) = -K_{\mu\nu}(Q) + \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\left(K_{\alpha\beta} - r_{\alpha\beta}\right) + r_{\mu\nu}. \tag{69}$$

The second equality becomes an identity upon insertion of the Legendre map. It may be verified that this expression coincides with that arising from the definition of the stress tensor, hence it is valid not only for solutions. As a consequence, the Einstein field equations turn into an identity when $g_{\mu\nu}$ and $\psi_{\mu\nu}$ are related by the Legendre transformation (56). From the Lagrange equations of motion for $\psi_{\mu\nu}$ (see below) it will be evident that the use of the equations does not simplify the formula for $T_{\mu\nu}$. Since $Q^\alpha_{\mu\nu}$ contains $\gamma^{\mu\nu}$, the energy–momentum tensor is a highly nonlinear function of $\psi_{\mu\nu}$; in general it also comprises linear terms. The presence of second order derivatives of $\psi_{\mu\nu}$ in $T_{\mu\nu}$ means that the energy density is not determined by initial data on a Cauchy surface. This is a generic feature of the stress tensors for vector and tensor fields: the vector gauge fields (represented by one–forms) and scalar fields are the only exceptions.

The two metrics are subject to Bianchi identities. That for $g_{\mu\nu}$, $G^{\mu\nu}{}_{;\nu} = 0$, is of less practical use while the other one, $\tilde{\nabla}_\nu \tilde{G}^\nu_\mu(\psi) = 0$, may be reformulated by setting $\tilde{R}_{\mu\nu} = r_{\mu\nu}(g, \psi)$ and replacing $\tilde{\nabla}_\nu$ by ∇_ν according to $\tilde{\nabla}_\nu A^\alpha = \nabla_\nu A^\alpha + Q^\alpha_{\nu\sigma}A^\sigma$ for any vector field. In this way the identity is transformed into first order equations for $r_{\mu\nu}$,

$$\gamma^{\alpha\beta}\left(r_{\mu\alpha;\beta} - \frac{1}{2}r_{\alpha\beta;\mu} - Q^\sigma_{\alpha\beta}r_{\mu\sigma}\right) = 0. \tag{70}$$

These constitute four differential constraints on $\psi_{\mu\nu}$; they are independent of the equations of motion for the field.

The Lagrange equations of motion are derived in the standard way in a simple but long and strenuous computation. These read

$$\begin{aligned}
E^{\mu\nu} &\equiv \nabla_\alpha\left(\frac{\partial(2L_M)}{\partial\psi_{\mu\nu;\alpha}}\right) - \frac{\partial(2L_M)}{\partial\psi_{\mu\nu}} = \gamma^{\alpha\beta}\left(K^{(\mu}{}_{\alpha\beta}{}^{\nu)} + Q^{(\mu\;;\nu)}_{\alpha\beta} - \frac{1}{2}g^{\mu\nu}Q^\sigma_{\sigma\alpha;\beta}\right) \\
&\quad -\gamma^{\alpha(\mu}g^{\nu)\beta}K_{\alpha\beta} - \gamma^{\alpha(\mu}Q^{\nu)\;;\beta}_{\alpha\beta} + \frac{1}{2}\gamma^{\mu\nu}g^{\alpha\beta}Q^\sigma_{\alpha\beta;\sigma} - 2\gamma^{\alpha\beta}Q^\sigma_{\sigma\alpha}Q^{(\mu}_{\beta\tau}g^{\nu)\tau} \\
&\quad +g^{\alpha\beta}\left(Q^\sigma_{\sigma\tau}Q^{(\mu}_{\alpha\beta}\gamma^{\nu)\tau} - Q^\sigma_{\alpha\tau}Q^{(\mu}_{\sigma\beta}\gamma^{\nu)\tau} + Q^{(\mu}_{\alpha\sigma}Q^{\nu)}_{\tau\beta}\gamma^{\sigma\tau}\right) \\
&\quad +\frac{1}{2}g^{\mu\nu}Q^\beta_{\alpha\beta}\left(\gamma^{\alpha\tau}Q^\sigma_{\sigma\tau} + \gamma^{\sigma\tau}Q^\alpha_{\sigma\tau}\right) - P^{\mu\nu} \equiv M^{\mu\nu}(\nabla\psi) - P^{\mu\nu} = 0, \tag{71}
\end{aligned}$$

where $M^{\mu\nu}(\nabla\psi)$ denotes the kinetic part of the equations comprising all derivative terms and $P^{\mu\nu}$ is the potential part,

$$P^{\mu\nu} \equiv \frac{\partial}{\partial\psi_{\mu\nu}}\left[-g^{\alpha\beta}r_{\alpha\beta}(g,\psi) + \left|\frac{\psi}{g}\right|^{1/2} f(\psi_{\alpha\beta}, r_{\alpha\beta}(g,\psi))\right]. \tag{72}$$

The kinetic part is *universal* while $P^{\mu\nu}$ explicitly depends on f and $r_{\alpha\beta}$. At first sight these equations seem intractably complicated but we shall see that for the special Lagrangian (33) they may be simplified and some interesting information can be extracted as well as a simple solution can be found.

Since in general $\psi_{\mu\nu}$ is a mixture of spin–0 and spin–2 fields, there are only four constraints (70). Only after extracting and removing the scalar field one can derive a fifth constraint. In the case of the linear massive spin–2 field in flat spacetime the fifth constraint is generated by the trace of the equations of motion [17] and something analogous occurs in the present case though the procedure is more involved. The trace $E^{\mu\nu}g_{\mu\nu} = 0$ is a second order equation while that with respect to $\psi_{\mu\nu}$ is (after many manipulations)

$$E^{\mu\nu}\psi_{\mu\nu} = \gamma^{\mu\nu}(J_{\mu;\nu} - J_{\mu}\psi_{\nu\alpha;\beta}\gamma^{\alpha\beta}) - P^{\mu\nu}\psi_{\mu\nu} = 0, \tag{73}$$

where $J_{\mu} \equiv \psi_{\mu\alpha}{}^{;\alpha} - \frac{1}{4}\tau\gamma^{\alpha\beta}\psi_{\alpha\beta;\mu}$ and $\tau \equiv g^{\mu\nu}\psi_{\mu\nu}$. We will see in the next section that J_{μ} vanishes in the particular case (33), inducing an additional constraint.

6   Constraints and the ground state solution in EF

From now on we restrict our study to the special quadratic Lagrangian (33) and expect that $\psi_{\mu\nu}$ will turn out to be a massive purely spin–2 non-geometric component of the gravitational doublet. The Legendre transformation (57) takes the form

$$g^{\mu\nu} = A[(1 + \frac{2}{3m^2}\tilde{R})\gamma^{\mu\nu} - \frac{2}{m^2}\tilde{R}^{\mu\nu}] \tag{74}$$

with

$$A \equiv \left|\frac{\psi}{g}\right|^{1/2} = |\det(\psi^{\alpha}_{\beta})|^{1/2}.$$

Inverting this transformation one gets

$$\tilde{R}_{\mu\nu}(\psi) = r_{\mu\nu}(g,\psi) = \frac{m^2}{2A}[(\tau - 3A)\psi_{\mu\nu} - \psi_{\mu\alpha}\psi^{\alpha}_{\nu}] \tag{75}$$

and

$$\tilde{R} = \gamma^{\mu\nu}r_{\mu\nu} = \frac{3m^2}{2A}(\tau - 4A).$$

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The Lagrangian (68) reads now

$$L_M = \frac{1}{2}K(Q) + \frac{m^2}{8A}(-\tau^2 + \psi^{\mu\nu}\psi_{\mu\nu} + 6A\tau - 12A^2). \tag{76}$$

Its potential part generates the potential piece of the equations of motion (71),

$$P^{\mu\nu} = \frac{m^2}{8A}\left[\left(\tau^2 - \psi^{\alpha\beta}\psi_{\alpha\beta} - 12A^2\right)\gamma^{\mu\nu} + 4\psi^{\mu\nu} + 4\left(3A - \tau\right)g^{\mu\nu}\right] \tag{77}$$

with the trace $P^{\mu\nu}\psi_{\mu\nu} = \frac{3}{2}m^2(\tau - 4A) = A\gamma^{\mu\nu}r_{\mu\nu}$.

Inserting $r_{\mu\nu}$ from eq. (75) into the constraints (70) and employing an identity valid for any two nonsingular symmetric tensors $\psi_{\mu\nu}$ and $g_{\mu\nu}$,

$$A_{,\mu} \equiv \frac{1}{2}A\gamma^{\alpha\beta}\psi_{\alpha\beta;\mu}, \tag{78}$$

one arrives at $J_\mu = 0$ for J_μ appearing in eq. (73). The constraints $J_\mu = 0$, which are equivalent to (70), hold independently of the equations of motion. Assuming that the equations $E^{\mu\nu} = 0$ hold one gets from (73) that $E^{\mu\nu}\psi_{\mu\nu} = 0 = -P^{\mu\nu}\psi_{\mu\nu}$ and this implies that $\tau = 4A$ or $\tilde{R}(\psi) = 0$. Vanishing of $\tilde{R}$ is evident from Einstein field equations (39) in HJF. Next inserting $A = \tau/4$ in (78) and simplifying the resulting equation by means of $J_\mu = 0$ one gets four differential constraints

$$\tau_{,\mu} - 2\psi_{\mu\alpha}{}^{;\alpha} = 0. \tag{79}$$

Together with the algebraic constraint $\tau = 4A$ they form five constraints imposed on $\psi_{\mu\nu}$ ensuring that the field has exactly five degrees of freedom and carries spin two. As in HJF, the gravitational doublet consists of spin–two fields, a massless (the metric) and a massive one [12, 7, 13, 14], clearly its mass is the same in both frames.

The algebraic constraint reduces (75) and (77) to

$$P^{\mu\nu} = \frac{m^2}{2\tau}[(\frac{1}{4}\tau^2 - \psi^{\alpha\beta}\psi_{\alpha\beta})\gamma^{\mu\nu} + 4\psi^{\mu\nu} - \tau g^{\mu\nu}] \tag{80}$$

and

$$r_{\mu\nu} = \frac{m^2}{2\tau}(\tau\psi_{\mu\nu} - 4\psi_{\mu\alpha}\psi_\nu^\alpha). \tag{81}$$

Now one can find a simple relationship between the massive fields in both the frames. $\pi^{\mu\nu}$ as a function of the EF variables is (from (74) and (39))

$$\pi^{\mu\nu}(g_{\alpha\beta}, \psi_{\alpha\beta}) = |\det(\psi_\beta^\alpha)|^{-1/2}g^{\mu\nu} - \gamma^{\mu\nu}, \tag{82}$$

we stress that $\gamma^{\mu\nu}$ is the inverse matrix to $\psi_{\mu\nu}$ and all indices (including these at $\psi_{\mu\nu}$) are shifted with the aid of $g_{\mu\nu}$.

By definition $\psi_{\mu\nu}$ is a nonsingular matrix and as such it is not the most suitable description of a classical field which should vanish in a ground state. A field redefinition is required. To this aim we first determine a ground state solution. In flat spacetime a ground state solution (vacuum) is Lorentz invariant, in a curvd one it should be covariantly constant, $\psi_{\alpha\beta;\mu} = 0$. This implies

$$Q^{\alpha}_{\mu\nu} = 0, \qquad \tau = \text{const} \qquad \text{and} \qquad \psi_{\alpha\beta}\psi^{\alpha\beta} = \text{const.} \qquad (83)$$

The equations $E^{\mu\nu} = 0$ reduce for this solution to $P^{\mu\nu} = 0$ and these read

$$4\psi_{\mu\alpha}\psi^{\alpha}_{\nu} = \tau\psi_{\mu\nu} + (\psi_{\alpha\beta}\psi^{\alpha\beta} - \frac{1}{4}\tau^2)g_{\mu\nu}. \qquad (84)$$

By inserting (84) into (81) one finds

$$r_{\mu\nu} = \frac{m^2}{8\tau}(\tau^2 - 4\psi_{\alpha\beta}\psi^{\alpha\beta})g_{\mu\nu} \equiv Cg_{\mu\nu} \qquad (85)$$

and the energy–momentum tensor is reduced to its potential part equal to $T_{\mu\nu} = -Cg_{\mu\nu}$. The ground state spacetime satisfies $G_{\mu\nu} = -Cg_{\mu\nu}$ and should be maximally symmetric, i.e. Minkowski, dS or AdS depending on the sign of C. Usually for a classical field its stress $T_{\mu\nu}$ vanishes in the ground state; for $\psi_{\mu\nu}$ this amounts to $\tau^2 = 4\psi_{\alpha\beta}\psi^{\alpha\beta}$ and $r_{\mu\nu} = 0$. Multiplying (81) by $\gamma^{\nu\sigma}$ one gets $\psi_{\mu\nu} = \frac{1}{4}\tau g_{\mu\nu}$ and $g_{\mu\nu} = \eta_{\mu\nu}$. Then $A = (\tau/4)^2$ and the constraint $\tau = 4A$ yields $\tau = 4$ as τ and A cannot vanish. Finally the ground state solution is $g_{\mu\nu} = \eta_{\mu\nu} = \psi_{\mu\nu}$.

Next assume that $C \neq 0$ and the spacetime is dS or AdS. Any covariantly constant $\psi_{\mu\nu}$ can be written as $\psi_{\mu\nu} = \frac{1}{4}\tau g_{\mu\nu} + \phi_{\mu\nu}$ with $\tau = \text{const}$, $g^{\mu\nu}\phi_{\mu\nu} = 0$ and $\phi_{\mu\nu;\alpha} = 0$. Inserting the Riemann tensor for these two spacetimes into the identity $R^{\sigma}{}_{\alpha\mu\nu}\phi_{\sigma\beta} + R^{\sigma}{}_{\beta\mu\nu}\phi_{\sigma\alpha} = 0$ which follows from Ricci identity for a covariantly constant $\phi_{\mu\nu}$, one finds $\phi_{\mu\nu} = 0$ — dS and AdS do not admit a traceless covariantly constant $\phi_{\mu\nu}$. Thus we have shown that the theory in EF has a unique ground state solution $g_{\mu\nu} = \eta_{\mu\nu} = \psi_{\mu\nu}$. This result was previously found in [13] under the simplifying assumption that the spin–two field is proportional to the metric $g_{\mu\nu}$; the proof here is generic. Of course there are many Einstein spaces in which $\psi_{\mu\nu;\alpha} = 0$, e.g. setting $\psi_{\mu\nu} = g_{\mu\nu}$ one gets $R_{\mu\nu}(g) = 0$ and conversely, for $\psi_{\mu\nu;\alpha} = 0$ and $R_{\mu\nu} = 0$ the only solution is $\psi_{\mu\nu} = g_{\mu\nu}$. However these spacetimes are not maximally symmetric and cannot be regarded as ground state solutions.

Hindawi et al. [13] have shown stability of the ground state solution against restricted scalar perturbations of the field $\psi_{\mu\nu}$. However to prove stability of the solution one should show either that arbitrary tensor excitations of the field do not grow in time,

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or equivalently that these generic fluctuations have positive energy density. The second way is impractical taking into account the complexity of the stress tensor (69): we will return to the problem of energy density for small excitations in a forthcoming paper. The ground state solution is mapped by the Legendre transformation (56) onto the $\psi_{\mu\nu} = \eta_{\mu\nu}$ and $\pi^{\mu\nu} = 0$ solution in HJF (sect. 3). The plane–wave perturbations of the solution in HJF can be transformed with the aid of the map (82) onto similar perturbations in EF showing the linear stability of the vacuum in the latter frame.

The massive spin–2 field should be described as an excitation above the ground state. In order to have a covariant description one makes a redefinition $\psi_{\mu\nu} \equiv g_{\mu\nu} + \Phi_{\mu\nu}$ and assumes that the gravitational doublet is EF=$\{g_{\mu\nu}, \Phi_{\mu\nu}\}$[10]. In terms of $\Phi_{\mu\nu}$ one has

$$Q_{\mu\nu}^\alpha = \frac{1}{2}\gamma^{\alpha\beta}(\Phi_{\beta\mu;\nu} + \Phi_{\beta\nu;\mu} - \Phi_{\mu\nu;\beta}), \qquad \Phi \equiv g^{\mu\nu}\Phi_{\mu\nu},$$

furthermore $\tau = \Phi + 4$ and the constraints read

$$\Phi + 4 = 4|\det(\delta_\beta^\alpha + \Phi_\beta^\alpha)|^{1/2}, \qquad \Phi_{\mu\alpha}{}^{;\alpha} - \frac{1}{2}\Phi_{,\mu} = 0. \tag{86}$$

The field equations explicitly expressed in terms of $\Phi_{\mu\nu}$ are given in Appendix.

7 Internal structure of the theory in EF

Consistency of the field equations in EF follows from the consistency in HJF and dynamical equivalence of the two frames. Alternatively, one can prove it directly in EF using, as previously, the coordinate invariance. Now the proof is slightly different. Under an infinitesimal transformation $x'^\mu = x^\mu + \varepsilon^\mu$ one finds $\delta\Phi_{\mu\nu} = 2\eta_{(\mu;\nu)} + 2\varepsilon_\alpha\Omega_{\mu\nu}{}^\alpha$, where $\eta_\mu \equiv -\Phi_{\mu\alpha}\varepsilon^\alpha$ and $\Omega_{\mu\nu}{}^\alpha \equiv \frac{1}{2}(\Phi^\alpha{}_{\mu;\nu} + \Phi^\alpha{}_{\nu;\mu} - \Phi_{\mu\nu}{}^{;\alpha})$. Using the definition (71) in $\delta\int d^4x\sqrt{-g}L_M = 0$ one arrives at an analogous strong Noether conservation law,

$$T^{\mu\nu}{}_{;\nu} - E^{\alpha\beta}{}_{;\beta}\Phi_\alpha^\mu - \Omega_{\alpha\beta}{}^\mu E^{\alpha\beta} = 0. \tag{87}$$

The energy–momentum tensor has a purely geometric origin, i.e. setting $\tilde{R}_{\mu\nu} = r_{\mu\nu}$ in (69) one gets $T_{\mu\nu} = G_{\mu\nu}$, thus its divergence always vanishes, $T^{\mu\nu}{}_{;\nu} = 0$. Consistency then requires that $E^{\alpha\beta}{}_{;\beta} = 0$ identically if $E^{\alpha\beta} = 0$ and the five constraints hold in the spacetime. However eqs. (71) are too complicated to allow for a direct check of the identity. This will be done in a subsequent paper in a perturbative analysis about the ground state solution up to the second order.

Also the problem of whether the Lagrange equations for $\Phi_{\mu\nu}$ are all hyperbolic propagation ones is harder in this frame. Here one should distinguish between eqs. (71) and

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[10]Tomboulis [14] employs a different decomposition of $\psi_{\mu\nu}$ into $g_{\mu\nu}$ and the massive spin–two field.

(102) since the latter arise from the former upon using the constraints. After long computations one finds that all eqs. (71) contain second time derivatives $\Phi_{\mu\nu,00}$; this is due to the nonlinearities. It is difficult to establish whether these equations are hyperbolic. We will return to the problem in a subsequent paper where it will be shown that up to the second order in an perturbation expansion these form a nondegenerate system of hyperbolic propagation equations (of Klein–Gordon type).

At present we investigate the structure of the linearized theory. Let a weak field $\Phi_{\mu\nu}$ be the source of some metric $g_{\mu\nu}$. One should not a priori assume that $g_{\mu\nu}$ is a fixed spacetime background for a small excitation $\Phi_{\mu\nu}$ since $T_{\mu\nu}(g, \Phi)$ may contain linear terms and the metric will then be affected by the excitations. The linearized form of eqs. (71) reads

$$
\begin{aligned}
2E^{(1)}_{\mu\nu} &= g_{\mu\nu}\left(\Phi^{\alpha\beta}{}_{;\alpha\beta} - \Box\Phi\right) + \Box\Phi_{\mu\nu} - R^{\alpha}{}_{(\mu}\Phi_{\nu)\alpha} - \Phi^{\alpha}{}_{\mu;\alpha\nu} - \Phi^{\alpha}{}_{\nu;\alpha\mu} \\
&\quad + 2R_{\mu\alpha\nu\beta}\Phi^{\alpha\beta} + \Phi_{;\mu\nu} - m^2\left(\Phi_{\mu\nu} - \Phi g_{\mu\nu}\right) = 0,
\end{aligned} \tag{88}
$$

here we have used $A \approx 1 + \frac{1}{2}\Phi$. The algebraic constraint gives in the linear approximation $\Phi = 0$ and the differential ones reduce to $\Phi_{\mu\nu}{}^{;\nu} = 0$. Applying these constraints one gets the linearized version of eqs. (102) which one denotes by $E^{\mathrm{L}}_{\mu\nu} = 0$. It turns out that the linearized energy–momentum tensor (101) is $T^{\mathrm{L}}_{\mu\nu} = E^{\mathrm{L}}_{\mu\nu}$, thus for the linearized equations of motion (101–102) one gets $T^{\mathrm{L}}_{\mu\nu} = 0$ and $R_{\mu\nu}(g) = 0$ (thus $\Phi_{\mu\nu}$ is actually decoupled from the metric which becomes a background) and finally $E^{\mathrm{L}}_{\mu\nu}$ is simplified to

$$
2E^{\mathrm{L}}_{\mu\nu} = \Box\Phi_{\mu\nu} - m^2\Phi_{\mu\nu} + 2R_{\mu\alpha\nu\beta}\Phi^{\alpha\beta} = 0; \tag{89}
$$

clearly $E^{(1)}_{\mu\nu}$ coincides with $E^{\mathrm{L}}_{\mu\nu}$ upon employing the constraints. Eq. (89) is identical to the linearized form of eq. (40) for $\pi^{\mu\nu}$ in HJF upon replacing $g_{\mu\nu}$ by $\psi_{\mu\nu}$. This follows from eq. (82) where one puts $A \approx 1$ and $\gamma^{\mu\nu} \approx g^{\mu\nu} - \Phi^{\mu\nu}$. Then $\Phi^{\mu\nu} \approx \pi^{\mu\nu}$ and in this approximation one may replace ∇_μ by $\tilde{\nabla}_\mu$.

Now one should show that the constraints $\Phi = 0 = \Phi_{\mu\nu}{}^{;\nu}$ (which for the moment will be referred to as secondary constraints) are preserved in time by the linear equations (89) and $R_{\mu\nu}(g) = 0$. To this end one determines which equations of the system (88) are not propagation ones. Using the Gauss normal coordinates in which $g_{00} = -1$ and $g_{0i} = 0$, $i = 1, 2, 3$, one finds that the four equations $E^{(1)}_{0\mu} = 0$ (and equivalently $E^{(1)0\mu} = 0$ and $E^{(1)0}_{\mu} = 0$) do not contain the time derivatives $\Phi_{\mu\nu,00}$; these will be referred to as primary constraints. Hence eqs. (88) form a degenerate system consisting of 6 propagation equations $E^{(1)}_{ik} = 0$ and the four primary constraints. This allows one to prove a proposition:

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If the propagation equations (89) and $R_{\mu\nu}(g) = 0$ hold throughout the spacetime and the following constraints restrict initial data on a given Cauchy surface:

- *the four primary constraints $E^{(1)}_{0\mu} = 0$,*

- *the five secondary constraints $\Phi = 0 = \Phi_{\mu\nu}{}^{;\nu}$ and*

- *additionally $\Phi_{,0} = 0$,*

then all the constraints, $\Phi = 0 = \Phi_{\mu\nu}{}^{;\nu}$ and $E^{(1)}_{0\mu} = 0$, are preserved in time and the eqs.(89) are equivalent to (88).

Proof: Let the Cauchy surface have a local equation $x^0 = 0$ in Gauss normal coordinates. The additional condition $\Phi_{,0} = 0$ at $x^0 = 0$ is essential since there are less primary constraints than the secondary ones.

a) The constraint $\Phi = 0$. A propagation equation for Φ is provided by taking the trace of eq. (89), $2g^{\mu\nu}E^{\mathrm{L}}_{\mu\nu} = \Box\Phi - m^2\Phi = 0$ since $R_{\alpha\beta} = 0$. The unique solution satisfying the initial conditions is $\Phi = 0$.

b) A propagation equation for the vector $S_\mu \equiv \Phi_{\mu\nu}{}^{;\nu}$ arises by taking divergence of eq. (89). In fact, in empty ($R_{\mu\nu} = 0$) spacetime the contraction of the full Bianchi identity yields $R_{\alpha\beta\mu\nu}{}^{;\nu} = 0$, then $\nabla^\nu\Box\Phi_{\mu\nu} = \Box S_\mu + 2R_{\sigma\mu\alpha\nu}\Phi^{\sigma\nu;\alpha}$ and $2E^{\mathrm{L}}_{\mu\nu}{}^{;\nu} = 0$ is reduced to the vector Klein–Gordon equation, $\Box S_\mu - m^2 S_\mu = 0$. Next applying $\Phi = 0$ one reexpresses the primary constraints in (88) in terms of S_μ, $2E^{(1)}_{0\mu} = g_{0\mu}S_\alpha{}^{;\alpha} - S_{0;\mu} - S_{\mu;0} = 0$ at $x_0 = 0$. In Gauss normal coordinates these read $S_{\mu,0} = 0$. The initial data $S_\mu = S_{\mu,0} = 0$ at $x_0 = 0$ imply then that $S_\mu \equiv 0$ for all times. This in turn implies vanishing of $E^{(1)}_{0\mu}$ in spacetime, furthermore eqs. (89) and (88) become equivalent everywhere.

As a byproduct it has been shown that the trace and divergence of eqs. (89) vanish identically giving rise to no further constraints besides the five secondary ones. This shows that the linear theory is fully consistent. It should be stressed that this theory is not identical with that for the linear massive test spin–2 field in empty spacetime, [17, 30], though both fields are subject to the same equations of motion, (89) and $R_{\mu\nu} = 0$, and to the same constraints. In fact, for the linear Fierz-Pauli spin–2 field $\Psi_{\mu\nu}$, it is known that all possible Lagrangians are equivalent (in flat spacetime) to the Wentzel Lagrangian

$$L_W(\Psi, m) \equiv \frac{1}{4}\left(-\Psi^{\mu\nu,\alpha}\Psi_{\mu\nu,\alpha} + 2\Psi^{\mu\nu,\alpha}\Psi_{\alpha\mu,\nu} - 2\Psi^{\mu\nu}{}_{,\nu}\Psi_{,\mu} + \Psi^{,\mu}\Psi_{,\mu}\right)$$
$$- \frac{m^2}{4}\left(\Psi^{\mu\nu}\Psi_{\mu\nu} - \Psi^2\right), \tag{90}$$

where $\Psi \equiv \eta^{\mu\nu}\Psi_{\mu\nu}$ [37, 17, 30, 39, 40]. In contrast, the linearized theory for the field $\Phi_{\mu\nu}$ above is not self–contained as a theory for a free spin–two field in a fixed background (in

particular, it has not its own Lagrangian) and arises only as a limit case of the nonlinear theory based on the Helmholtz Lagrangian (66) and (76).

The difference is most easily seen while dealing with energy density. The Wentzel Lagrangian generates a quadratic stress tensor $T_{\mu\nu}^W$ for the linear field [30, 40], while in the linearized theory for $\Phi_{\mu\nu}$ one finds that the lowest order terms of the expansion of the stress tensor are linear and vanish identically for solutions of the linearized field equations, while the quadratic part in the expansion of $T_{\mu\nu}$ (which is not presented here) does *not* coincide with $T_{\mu\nu}^W$.

This fact seems surprising, but one has to keep in mind that $T_{\mu\nu}^W$ is derived from L_W by replacing the fixed metric $\eta_{\mu\nu}$ by a generic metric $g_{\mu\nu}$ and the ordinary derivatives with covariant ones, then taking the variational derivative with respect to $g_{\mu\nu}$. In this derivation one regards the variation of the spin–two field to be zero, $\delta_g \Phi_{\mu\nu} \equiv 0$. On the contrary, when computing the stress tensor for the exact Lagrangian L_M (68), one regards the field $\psi_{\mu\nu}$ as fundamental, $\delta_g \psi_{\mu\nu} \equiv 0$, and since $\psi_{\mu\nu} \equiv g_{\mu\nu} + \Phi_{\mu\nu}$ then $\delta_g \Phi_{\mu\nu} = -\delta g_{\mu\nu}$. The fact that the spin–two field excitations are defined to vanish when $\psi_{\mu\nu}$ is equal to the actual spacetime metric $g_{\mu\nu}$ (and not to some fixed background metric, which would be physically objectionable and would also make the equations rather cumbersome) implies that the true stress tensor will contain both linear and quadratic terms not related to the Wentzel Lagrangian. Furthermore, even assuming $\delta_g \Phi_{\mu\nu} = -\delta g_{\mu\nu}$ while taking the variation of L_W would not be enough to recover the quadratic terms of the true stress tensor, because the higher order terms in the expansion of L_M do contribute to quadratic terms in the stress tensor. The case of the linearized theory for $\Phi_{\mu\nu}$ clearly shows that in general the equations of motion alone are insufficient for determining an energy–momentum tensor for a given field [38].

Finally we find the expression for the Helmholtz Lagrangian (66) and (76) in the lowest order approximation around the ground state solution $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Phi_{\mu\nu} = 0$ (where *both* metric and massive field perturbations are now taken into account). The metric and massive field excitations are $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ and $\Phi_{\mu\nu} = \epsilon \varphi_{\mu\nu} + \epsilon^2 \xi_{\mu\nu}$. Expanding the gravitational part one finds $R(g)\sqrt{-g} \approx \epsilon^2 L_W(h, 0)$ (the Wentzel Lagrangian is in fact known to coincide with the lowest–order expansion of the Einstein–Hilbert Lagrangian), and similarly for $2L_M$ in eq. (76), so that up to a full divergence (some auxiliary expressions are given in Appendix)

$$ L_H(\eta_{\mu\nu} + \epsilon h_{\mu\nu}, \epsilon \varphi_{\mu\nu} + \epsilon^2 \xi_{\mu\nu}) \sqrt{-\det(\eta_{\mu\nu} + \epsilon h_{\mu\nu})} \approx \epsilon^2 \left[L_W(h, 0) - L_W(\varphi, m) \right]. \quad (91) $$

Up to the second order the fields are decoupled (there are no interaction terms); also the field $\xi_{\mu\nu}$ is absent. In this way we have rederived the well known fact that in the linear approximation the massive spin–two field $\varphi_{\mu\nu}$ arising from a nonlinear gravity theory is a ghost field (a "poltergeist") [16, 25, 12, 14, 13, 18] (a full family of unphysical ghost fields appears in the linearized theory according to a different approach in [41]). This outcome is inescapable since Wald [42] gave a generic argument that in any generally

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covariant theory of a number of consistently interacting spin–two fields at least one field is necessarily ghostlike. Thus consistency of gravitational interactions implies that the massive spin–two field produces states of negative norm in the state vector space of quantum theory.

8   Exact solutions in Einstein frame

The field equations (101–102) are fairly involved, nevertheless one may seek for non-trivial solutions. It is well known (cf. e.g. [16]) that the corresponding fourth–order field equations for the metric $\tilde{g}_{\mu\nu}$ in JF admit solutions $\tilde{R}_{\mu\nu}(\tilde{g}) = 0$ which are trivial in the sense that they are vacuum solutions already in Einstein's general relativity. We therefore regard as nontrivial those solutions to eqs. (101–102) which after transforming back to JF yield nonvanishing Ricci tensor.

A general method for seeking solutions is to investigate known classes of geometrically distinguished metrics depending on some arbitrary functions and check if they can satisfy Einstein field equations (101) for a suitably chosen field $\Phi_{\mu\nu}$. If solutions are not precluded (see below) one may attempt to solve the entire system (101–102). Clearly some simplifying assumptions regarding $\Phi_{\mu\nu}$ are indispensable. It turns out that the constraints (86) are very stringent and one should solve them for a given class of $\Phi_{\mu\nu}$ and $g_{\mu\nu}$ before dealing with the field equations.

A generic simplifying assumption is to represent the tensor $\Phi_{\mu\nu}$ in terms of few scalar or vector functions. The simplest ansatz, that $\Phi_{\mu\nu}$ is completely described by its trace, $\Phi_{\mu\nu} = \frac{1}{4}\Phi g_{\mu\nu}$, is excluded by the constraints. In fact, the algebraic constraint (86) is then easily solved yielding $\Phi = -4$ or $\Phi = 0$. The first solution is excluded because it would correspond to vanishing of the original metric, $\tilde{g}_{\mu\nu} = 0$, while the second is trivial (in the above sense) since for $\Phi_{\mu\nu} = 0$ one has $R_{\mu\nu}(g) = \tilde{R}_{\mu\nu}(\tilde{g}) = 0$.

A static spherically symmetric solution is not known. Then we consider the simpler case of a metric representing a plane–fronted gravitational wave with parallel rays (a *pp wave*) [43, 44]. Such waves are characterized by a null covariantly constant wave vector (ray) k_{μ}, $k^{\mu}k_{\mu} = 0$ and $k_{\mu;\nu} = 0$. First we assume that the spin–two field is of the form $\Phi_{\mu\nu} = 2k_{(\mu}W_{,\nu)}$, where W is some scalar function, and it is null in the sense that $\Phi_{\mu\nu}k^{\nu} = 0$. The latter condition holds if $k^{\mu}W_{,\mu} = 0$, what implies that the gradient $W_{,\mu}$ is either null and proportional to k_{μ}, or it is spacelike and orthogonal to k_{μ}. If we consider the spacelike case, $W^{,\mu}W_{,\mu} > 0$, we immediately get the trace $\Phi = 0$ and then the algebraic constraint holds identically for any scalar W. The four differential constraints (86) reduce to the equation $\Box W = 0$. Then one finds that the expression of the stress tensor becomes

$$T_{\mu\nu} = -m^2 \left(k_{(\mu}W_{,\nu)} + \frac{1}{2}W^{,\alpha}W_{,\alpha}k_{\mu}k_{\nu} \right), \tag{92}$$

which is traceless. However, it is known [44] that Ricci tensor for a pp wave is proportional to $k_\mu k_\nu$. This condition is met by $T_{\mu\nu}$ as in (92) only if $W^{,\mu} W_{,\mu} = 0$, contrary to our assumption.

The other possibility is $\Phi_{\mu\nu} = w \, k_\mu k_\nu$ (which includes the special case of the previous ansatz where $W_{,\mu} = w k_\mu$). Once again $\Phi = 0$ and the algebraic constraint becomes an identity. The field $\Phi_{\mu\nu}$ is null by definition and the condition $k^\mu w_{,\mu} = 0$ arises now from the differential cosntraints. The inverse matrix is $\gamma^{\mu\nu} = g^{\mu\nu} - \Phi^{\mu\nu}$, while

$$Q^\alpha_{\mu\nu} = k^\alpha k_{(\mu} w_{,\nu)} - \frac{1}{2} w^{,\alpha} k_\mu k_\nu \tag{93}$$

implies

$$Q^\alpha_{\mu\nu;\alpha} = -\frac{1}{2} k_\mu k_\nu \Box w. \tag{94}$$

The energy–momentum tensor,

$$T_{\mu\nu} = \frac{1}{2} k_\mu k_\nu \left(\Box w - m^2 w \right), \tag{95}$$

is admissible by the pp–wave metric; one then evaluates the Lagrange equations of motion (102): these reduce to one scalar equation

$$\left(\Box - m^2 \right) w = 0, \tag{96}$$

hence $T_{\mu\nu} = 0$ and $R_{\mu\nu} = 0$. We have arrived at a rather surprising result: although the spacetime is not empty since $\Phi_{\mu\nu} \neq 0$, the metric satisfies the vacuum field equations and the spin–two field necessarily behaves as a test matter and carries no energy. To elucidate it one takes into account that a pp–wave metric represents its own linear approximation around flat spacetime [43, 45, 46]: the inverse matrix $g^{\mu\nu}$ equals its linearized version and the exact Einstein equations become linear and coincide with the linearized Einstein tensor. In this case linearity of the equations of motion and the assumption that the presence of the massive field preserves the pp–wave form of the metric imply that the interaction of the metric with its source is quite restricted. In fact, $\Phi_{\mu\nu} = w \, k_\mu k_\nu$ means that the massive field propagates in the same direction as the pp wave or that their momenta are parallel. In this situation the equations of motion and $T_{\mu\nu}$ are reduced to their linear parts. As we have seen in sect. 7, in the linearized theory $T^L_{\mu\nu} = 0$ by virtue of $E^L_{\mu\nu} = 0$ for any metric, not necessarily being a pp wave.

To complete the solution one solves (96) for w. Writing the pp–wave metric as in [44]

$$ds^2 = -2H(u, x, y)du^2 - 2du \, dv + dx^2 + dy^2, \tag{97}$$

where the hypersurfaces $u =$ const are null and are wave–fronts and $k_\mu = -\partial_\mu u$, one finds $\frac{\partial w}{\partial v} = 0$ and the Klein–Gordon eq. (96) is reduced to Yukawa equation on Euclidean plane,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - m^2 w = 0. \tag{98}$$

Solutions to the latter one are well known, and employing polar coordinates r and θ are expressed in terms of Bessel functions

$$
\begin{aligned}
w(u, r, \theta) \;=\;& \left[A_0(u) J_0(imr) + i B_0(u) H_0^{(1)}(imr) \right] \left[C_0(u)\theta + D_0(u) \right] \\
& + \sum_{n=1}^{\infty} \left[A_n(u) I_n(imr) + B_n(u) K_n(imr) \right] \left[C_n(u)\sin(n\theta) + D_n(u)\cos(n\theta) \right],
\end{aligned} \tag{99}
$$

here I_n and K_n are the modified Bessel functions, J_0 and $H_0^{(1)}$ are the usual Bessel and Hankel functions, while A_n, B_n, C_n and D_n $(n = 0, \ldots \infty)$ are arbitrary functions of the retarded time u.

It should be emphasized that the solution (95)–(99) is nontrivial in spite of $R_{\mu\nu} = 0$. To show it one employs eq. (74); first its trace w.r. to $\psi_{\mu\nu}$ yields $\tilde{R} = 0$, then the transformation is reversed to give

$$
\tilde{R}^{\mu\nu} = -\frac{m^2}{2} w \, k^\mu k^\nu. \tag{100}
$$

The ray vector is defined by $k_\mu = -\partial_\mu u$ and then $k^\mu = g^{\mu\nu} k_\nu = \gamma^{\mu\nu} k_\nu$, thus it is null in both metrics, $g^{\mu\nu} k_\mu k_\nu = \gamma^{\mu\nu} k_\mu k_\nu = 0$. This stems from the fact that $\psi_{\mu\nu}$ is a pp–wave metric too of the form (97), with the determining function $\tilde{H}(u, x, y) = H - \frac{1}{2}w$. From (97) one gets $R_{\mu\nu} = k_\mu k_\nu \Box H$, where $\Box H$ reduces to the 2–dimensional Laplacian as in (98) and $\Box H = 0$ for $R_{\mu\nu} = 0$. In Jordan frame one has $\tilde{\Box}\tilde{H} = \Box\tilde{H}$ and (100) follows. One checks directly that eq. (100) represents a solution of the fourth-order field equations. The ray vector is covariantly constant w.r. to $\psi_{\mu\nu}$ too, $\tilde{\nabla}_\mu k_\nu = 0$, since this property is independent of the metric functions H and $\tilde{H}$.

Clearly in HJF Ricci tensor is also given by eq. (100). Then, according to (35), $\pi_{\mu\nu} = w \, k_\mu k_\nu$ and its energy–momentum tensor is equal, by virtue of (36), to Ricci tensor. At first sight it seems better than in EF because here a non–vanishing field carries a non–vanishing energy. However a stress tensor of the form (100) describes a flux of energy and momentum moving at light velocity, as e.g. for a plane monochromatic electromagnetic wave. This is not the case of a massive field (particle) of any spin. This stress tensor in HJF is misleading and in this sense the vanishing of $T_{\mu\nu}$ in EF is closer to reality. However it should be emphasized that this bizarre situation is due to the peculiar choice $\Phi_{\mu\nu} = w \, k_\mu k_\nu$. One should compare the stress tensors in both frames for more physical solutions.

9   Conclusions

A model for a massive spin–two field consistently interacting with Einstein's gravity is provided by a nonlinear gravity theory, in particular by its simplest version with a Lagrangian quadratic in the Ricci tensor. The consistency is achieved at the cost of

introducing the nonlinearity. The model can be formulated as a second order Lagrangian field theory in many mathematically equivalent versions ("frames"). Among them two are distinguished: HJF and EF. In both frames the respective spacetime metric satisfies Einstein's field equations with the spin–two field acting as a matter source: this is in fact a result of the Legendre transformation, not of the choice of a particular frame.

Though the equations of motion in both frames are similar, the Lagrangian structure is different. While in EF the model has the standard form of classical field theory, in HJF the essential nonminimal coupling to curvature and the absence of a kinetic Lagrangian for the spin–two field indicate that the field should be viewed as a nongeometric member of the gravitational doublet (or triplet) rather that as ordinary matter.

The transformation between the two frames is akin (in a generalized sense) to the known canonical transformation of Hamiltonian mechanics which interchanges the roles of particles' positions and momenta, $(q, p) \mapsto (-p, q)$. In a similar way, one cannot assume that both the old and the new variables have the same physical interpretation as measurable quantities: in most cases one set of variables is in this sense unphysical. It is well known that for a quadratic Lagrangian (1) the total gravitational energy (in JF) is indefinite [47, 48] and in HJF the energy–momentum tensor for the massive spin–two field is unphysical as being purely linear. Thus HJF is unphysical in both senses, while at least in the case of restricted NLG theories ($L = f(\tilde{R})$) Einstein frame is physical since the positive energy theorem holds. Thus energy plays a crucial role in determining the physical (measurable) variables because energy density is qualitatively sensitive to the change of the spacetime metric between HJF and EF. Furthermore Einstein frame is advantageous in that it is unique while HJF is not.

It is not quite clear how harmful for classical nonlinear field theory is the fact that in the lowest order approximation the massive spin–two field is a ghost. The (exact) ground state solution is stable against linear perturbations. As regards energy, one cannot expect that the true expression for the energy–momentum tensor for the field (that in EF) is positive definite. That this is not the case follows from the existence of the solution found in sect. 8 for which $T_{\mu\nu} = 0$. This energy–momentum tensor violates the commonly accepted postulate that $T_{\mu\nu}$ vanishes if and only if the matter field vanishes [22]. Nevertheless one may still regard this model as a viable theory for a massive spin–two field. To do justice to the model one should study it in the second–order approximation. This will be done in a forthcoming paper.

Acknowledgements

Appendix

From the constraints (86) one derives $Q_{\mu\alpha}^\alpha = \Phi_{,\mu}(\Phi+4)^{-1}$ and $g^{\mu\nu}Q_{\mu\nu}^\alpha = 0$. Then Einstein field equations are

$$
\begin{aligned}
G_{\mu\nu}(g) &= T_{\mu\nu}(g,\Phi) = -Q_{\mu\nu;\alpha}^\alpha - Q_{\mu\nu}^\alpha \Phi_{,\alpha}(\Phi+4)^{-1} + Q_{\mu\beta}^\alpha Q_{\alpha\nu}^\beta - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}Q_{\alpha\tau}^\sigma Q_{\sigma\beta}^\tau \\
&\quad + \left(\Phi_{;\mu\nu} - \frac{1}{2}g_{\mu\nu}\Box\Phi\right)(\Phi+4)^{-1} - \left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right)(\Phi+4)^{-2} \quad (101) \\
&\quad + \frac{m^2}{2}\left[(\Phi-4)\Phi_{\mu\nu} - 4\Phi_{\mu\alpha}\Phi_\nu{}^\alpha - \frac{1}{2}g_{\mu\nu}\left(\Phi^2 - 2\Phi - 4\Phi_{\alpha\beta}\Phi^{\alpha\beta}\right)\right](\Phi+4)^{-1},
\end{aligned}
$$

this energy–momentum tensor is much simpler than that for a linear massive spin–two field in Minkowski space [30]. The Lagrange field equations read

$$
\begin{aligned}
E^{\mu\nu} &= \gamma^{\alpha(\mu}\nabla^{\nu)}\left(\Phi_{,\alpha}(\Phi+4)^{-1}\right) + \gamma^{\alpha\beta}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\alpha\left(\Phi_{,\beta}(\Phi+4)^{-1}\right) + 2\nabla_\alpha\left(\gamma^{\lambda(\mu}\Phi^{\nu)}{}_{[\lambda;\beta]}\right)\right. \\
&\quad \left. -2\gamma^{\tau\sigma}\gamma^{\lambda(\mu}\Phi^{\nu)}{}_{[\lambda;\tau]}\Phi_{\alpha\sigma;\beta} + \left(\frac{1}{2}g^{\mu\nu}\gamma^{\lambda\sigma}\Phi_{\alpha\lambda;\sigma} - \gamma^{\lambda(\mu}\Phi_{\alpha\lambda}{}^{;\nu)}\right)\Phi_{,\beta}(\Phi+4)^{-1}\right] \\
&\quad + \gamma^{\alpha(\mu}\gamma^{\nu)\beta}\left[-\frac{1}{2}\Box\Phi_{\alpha\beta} + \gamma^{\lambda\rho}\left(\frac{1}{2}\Phi_{\alpha\lambda}{}^{;\sigma}\Phi_{\beta\rho;\sigma} + 2\Phi^\sigma{}_{[\alpha;\lambda]}\Phi_{\sigma[\beta;\rho]}\right)\right] \\
&\quad - \frac{m^2}{2}\left[\left(\frac{1}{4}\Phi^2 - \Phi_{\alpha\beta}\Phi^{\alpha\beta}\right)\gamma^{\mu\nu} + 4\Phi^{\mu\nu} - \Phi g^{\mu\nu}\right](\Phi+4)^{-1} = 0, \quad (102)
\end{aligned}
$$

where $\Box\Phi_{\alpha\beta} \equiv \Phi_{\alpha\beta;\lambda}{}^{;\lambda}$.

In expanding the Helmholtz Lagrangian density $L_H\sqrt{-g}$ up to second order around the ground state solution the following expressions are useful,

$$
K(Q)\sqrt{-g} \approx -\epsilon^2 L_W(\varphi,0), \quad (103)
$$

$$
\Phi \approx \epsilon\varphi + \epsilon^2(\xi - h^{\mu\nu}\varphi_{\mu\nu}), \qquad \Phi^{\mu\nu}\Phi_{\mu\nu} \approx \epsilon^2\varphi^{\mu\nu}\varphi_{\mu\nu}, \quad (104)
$$

where $\varphi \equiv \eta^{\mu\nu}\varphi_{\mu\nu}$ and $\xi \equiv \eta^{\mu\nu}\xi_{\mu\nu}$,

$$
A \approx 1 + \frac{1}{2}\epsilon\varphi + \frac{1}{2}\epsilon^2\left(\frac{1}{4}\varphi^2 - \frac{1}{2}\varphi_{\alpha\beta}\varphi^{\alpha\beta} - \varphi_{\alpha\beta}h^{\alpha\beta} + \xi\right), \quad (105)
$$

$$
\tau \approx 4 + \epsilon\varphi + \epsilon^2\left(\xi - \varphi_{\alpha\beta}h^{\alpha\beta}\right) \quad (106)
$$

and

$$
\psi^{\alpha\beta}\psi_{\alpha\beta} \approx 4 + 2\epsilon\varphi + \epsilon^2\left(2\xi - 2\varphi_{\alpha\beta}h^{\alpha\beta} + \varphi_{\alpha\beta}\varphi^{\alpha\beta}\right). \quad (107)
$$

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